EE 637 Final

May 1, Spring 2007

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Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 40 points for a total score of 200 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\mathrm{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(40pt)

Let X_n be an i.i.d. random process with each X_n uniformally distributed on the interval $[0, 1]$.

a) Calculate $\mu = E[X_n]$ and $\sigma^2 = E[(X_n - \mu)^2]$.

Define $Y_n = X_n - \mu$.

b) Is Y_n a strict-sense stationary random process? Justify your answer.

c) Calculate the autocorrelation function $R_y(k) = E[Y_n Y_{n+k}]$.

Let $Z_n = Y_n * h_n$ where $h_n = \delta_n - \delta_{n-1}$

d) Calculate the autocorrelation function $R_z(k) = E[Z_n Z_{n+k}]$.

e) Calculate $S_z(\omega)$ the power spectrum of the random process Z_n.

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Problem 2.(40pt)

Consider an MRI that only images in one dimension, x. So for example, the object being imaged might be a thin rod oriented along the x-dimension.

In this example, assume that the magnetic field strength at each location is given by

$$M_o + G(t)x$$

where M_o is the static magnetic field strength and $G(t)x$ is the linear gradient field in the x dimension. Then the frequency of precession for a hydrogen atom (in rad/sec) is given by the product of γ, the gyromagnetic constant, and the magnetic field strength.

a) Calculate $\omega(x, t)$, the frequency of precession of a hydrogen atom at location x and time t.

b) Calculate $\phi(x, t)$, the phase of precession of a hydrogen atom at location x and time t assuming that $\phi(x, 0) = 0$.

c) Calculate $r(x, t)$, the signal radiated from hydrogen atoms in the interval $[x, x + dx]$ at time t.

d) Calculate $r(t)$, the signal radiated from hydrogen atoms along the entire object.

e) Calculate an expression for $a(x)$, the quantity of processing hydrogen atoms along the thin rod, from the function $r(t)$.

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Problem 3.(40pt)

Consider the 2-D error diffusion algorithm specified by the equations

$$\begin{aligned} z(m,n) &= Q\left[y(m,n)\right] \\ e(m,n) &= y(m,n) - z(m,n) \\ y(m,n) &= x(m,n) + h(m,n) * e(m,n) \end{aligned}$$

where (m,n) are the row and column respectively, $0 \leq x(m,n) \leq 1$ is the input, $z(m,n)$ is the output, $h(m,n) = \delta(m-1,n)$ is a strictly causal filter, and $Q[\cdot]$ is a quantizer with the form

$$Q[y] = \begin{cases} 1 & \text{if } y > 0.5 \\ 0 & \text{if } y \leq 0.5 \end{cases} .$$

a) Assuming that $e(m,n) = 0$ for $m < 0$ or $n < 0$ and that $x(m,n)$ is given by

$x(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	0.25	0.25	0.25	0.25	0.25
$n=1$	0.25	0.25	0.25	0.25	0.25
$n=2$	0.25	0.25	0.25	0.25	0.25

Then compute the modified input $y(m,n)$

$y(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$					
$n=1$					
$n=2$					

and compute the output $z(m,n)$.

$z(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$					
$n=1$					
$n=2$					

b) Assume that the quantizer error, $e(m,n)$, is white noise that is uniformly distributed on the interval $[-0.5, 0.5]$, and calculate the power spectrum for the random process $e(m,n)$.

c) Using the same assumptions as part b), compute the power spectrum of the display error $d(m,n) = z(m,n) - x(m,n)$.

d) Propose a "better" filter $h(m,n)$, and justify why you would expect your choice to produce a better quality image.

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Problem 4.(40pt)

A physical system measures the signal $y(t) = x(t) * h(t)$, where $h(t) = \mathrm{rect}(t)$; and you would like to recover the signal $x(t)$ from your measurements.

In order to approximately recover the signal $x(t)$, you filter the measured signal with a filter $g(t)$.

$$\hat{x}(t) = g(t) * y(t) \ .$$

Let $Y(f)$, $X(f)$, $\hat{X}(f)$, $H(f)$, and $G(f)$ denote the continuous-time Fourier transform of each signal and impulse response.

a) Is it possible to select $g(t)$ so that every frequency can be recovered from the original signal $x(t)$? If so, how? If not, what frequencies can not be recovered?

For the next parts, assume that we choose

$$G(f) = \frac{H^*(f)}{|H(f)|^2 + \epsilon}$$

where $H^*(f)$ is the complex conjugate of $H(f)$.

b) Show that for most frequencies, $\hat{X}(f) \approx X(f)$, as $\epsilon \to 0$.

c) What is the disadvantage of letting $\epsilon \to 0$?

d) How might one choose ϵ in a practical situation?

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Problem 5.(40pt)

Consider an signal $Y_n = S_n + W_n$ where S_n is a unknown signal, W_n is i.i.d. Gaussian noise with mean 0 and variance 1.

Your job (should you choose to accept it) is to recover a good estimate of S_n by applying a function to a 5 point window about the location n. So the estimate is given by

$$\hat{S}_n = f(Z_n)$$

where

$$Z_n = [Y_{n-2}, Y_{n-1}, Y_n, Y_{n+1}, Y_{n+2}]^t \ .$$

a) Assuming that $S_n = \mu$, where μ is a constant, then what is a good choice for the function $f(\cdot)$? Justify your answer.

b) Assuming that S_n is a slowly varying function of n, then what is a good choice for the function $f(\cdot)$? Justify your answer.

c) Assuming the S_n has the form

$$S_n = a_n + (10)b_n$$

where a_n is a slowly varying function of n, and b_n is i.i.d. with discrete probability density function $P\{b_n = 1\} = P\{b_n = -1\} = 0.001$ and $P\{b_n = 0\} = 0.998$, then what is a good choice for the function $f(\cdot)$? Justify your answer.

d) Assuming the S_n has the form

$$S_n = a_n + 10 \sum_{k=-\infty}^{\infty} b_k \text{pulse}_{10}(n-k)$$

where a_n is a very slowly varying function of n, and b_n is i. i. d. with discrete probability density function $P\{b_n = 1\} = P\{b_n = -1\} = 0.001$ and $P\{b_n = 0\} = 0.998$, then what is a good choice for the function $f(\cdot)$? [1] Justify your answer.

[1] Note that $\text{pulse}_N(n) = u(n) - u(n-N)$.

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