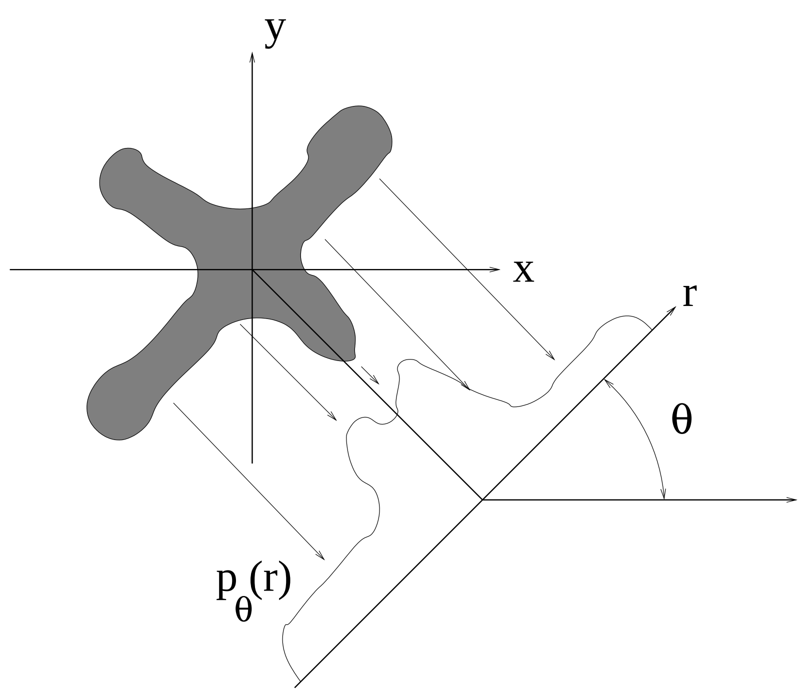
# Tomography

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.
  - Projections must be collected at every angle $\theta$ and displacement $r$.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.



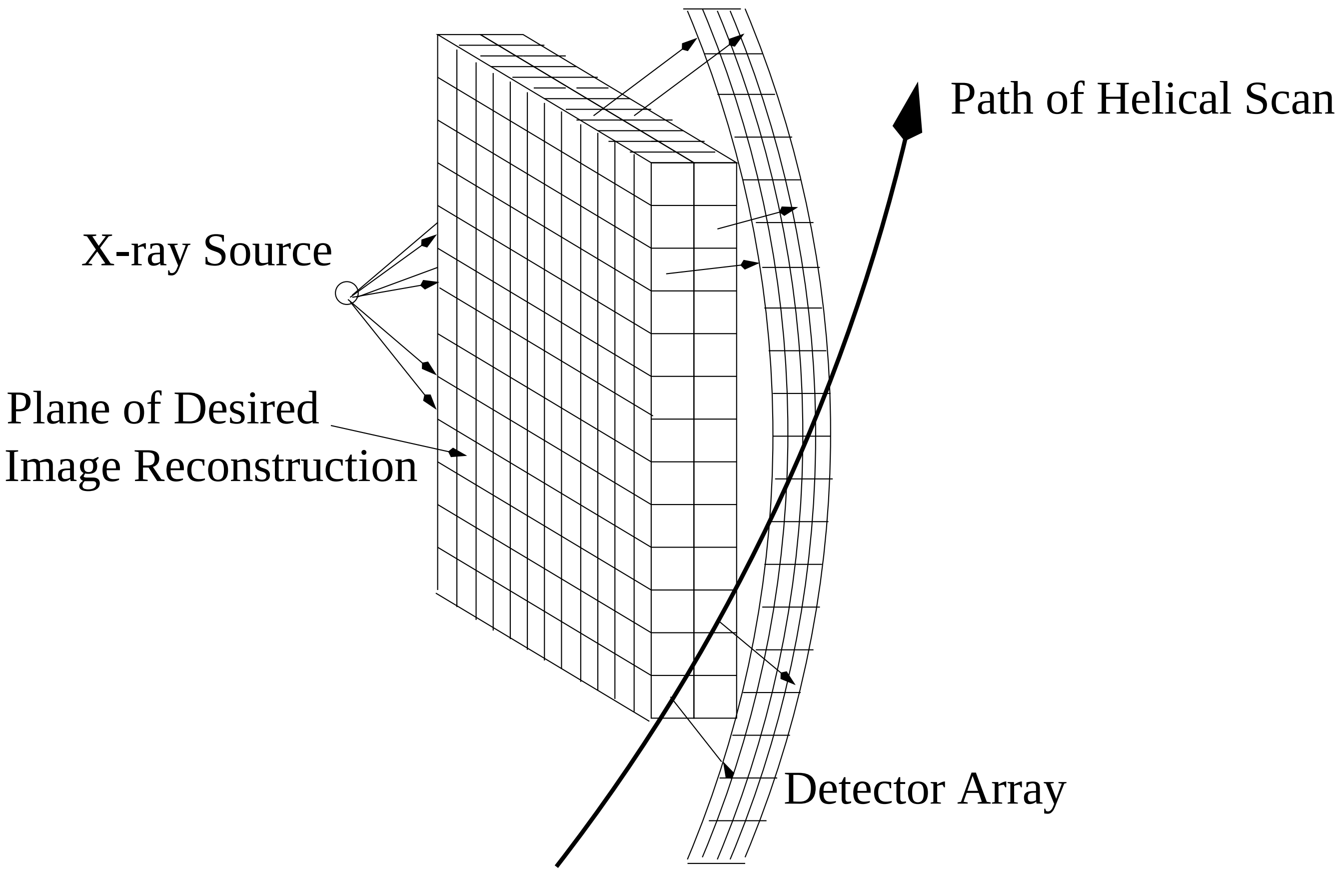

- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

# Medical Imaging Modalities

- Anatomical Imaging Modalities
  - Chest X-ray
  - Computed Tomography (CT)
  - Magnetic Resonance Imaging (MRI)
- Functional Imaging Modalities
  - Signal Photon Emission Tomography (SPECT)
  - Positron Emission Tomography (PET)
  - Functional Magnetic Resonance Imaging (fMRI)

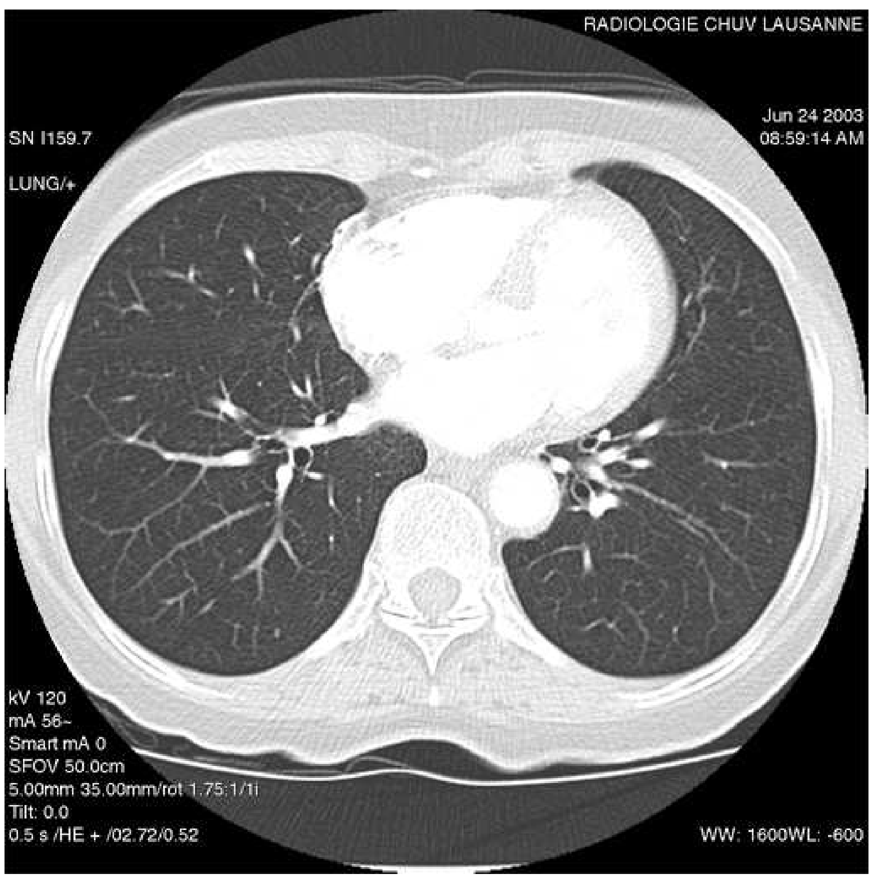
# Multislice Helical Scan CT

- Multislice CT has a cone-beam structure

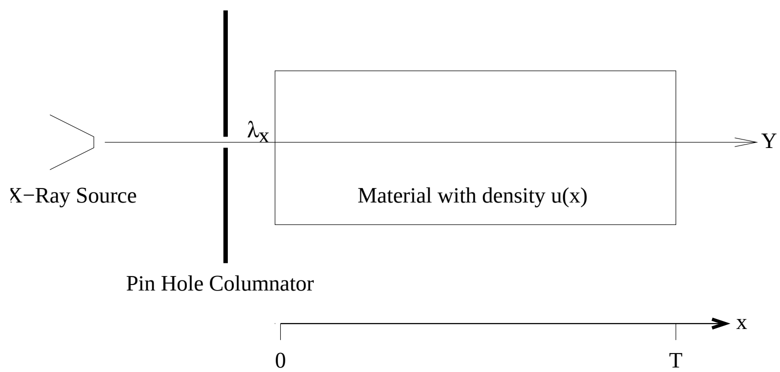


# Example: CT Scan

- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan



# Photon Attenuation




$x$ - depth into material measured in cm

$Y_x$ - Number of photons at depth $x$

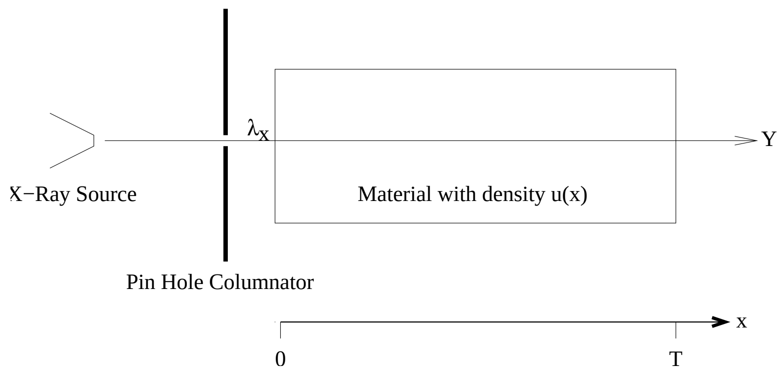
$\lambda_x = E[Y_x]$

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x}\lambda_x^k}{k!} .$$

- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.

# Differential Equation for Photon Attenuation




The attenuation of photons obeys the following equation

$$\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x$$

where $\mu(x)$ is the density in units of $\text{cm}^{-1}$.

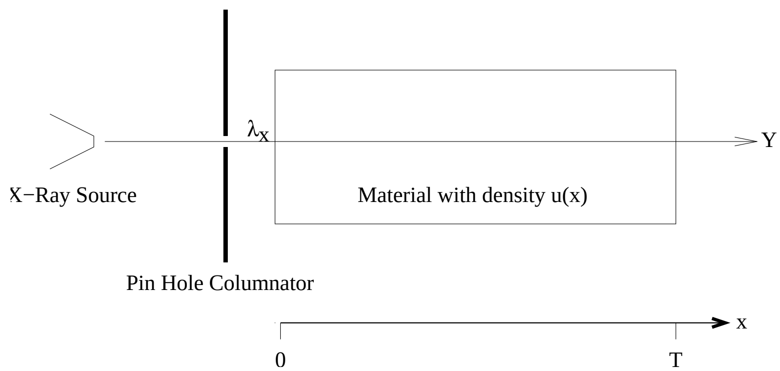
- The solution to this equation is given by

$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t)dt}$$

- So we see that

$$\begin{aligned} \int_0^x \mu(t)dt &= -\log\left(\frac{\lambda_x}{\lambda_0}\right) \\ &\approx -\log\left(\frac{Y_x}{\lambda_0}\right) \end{aligned}$$

# Estimate of the Projection Integral




A commonly used estimate of the projection integral is

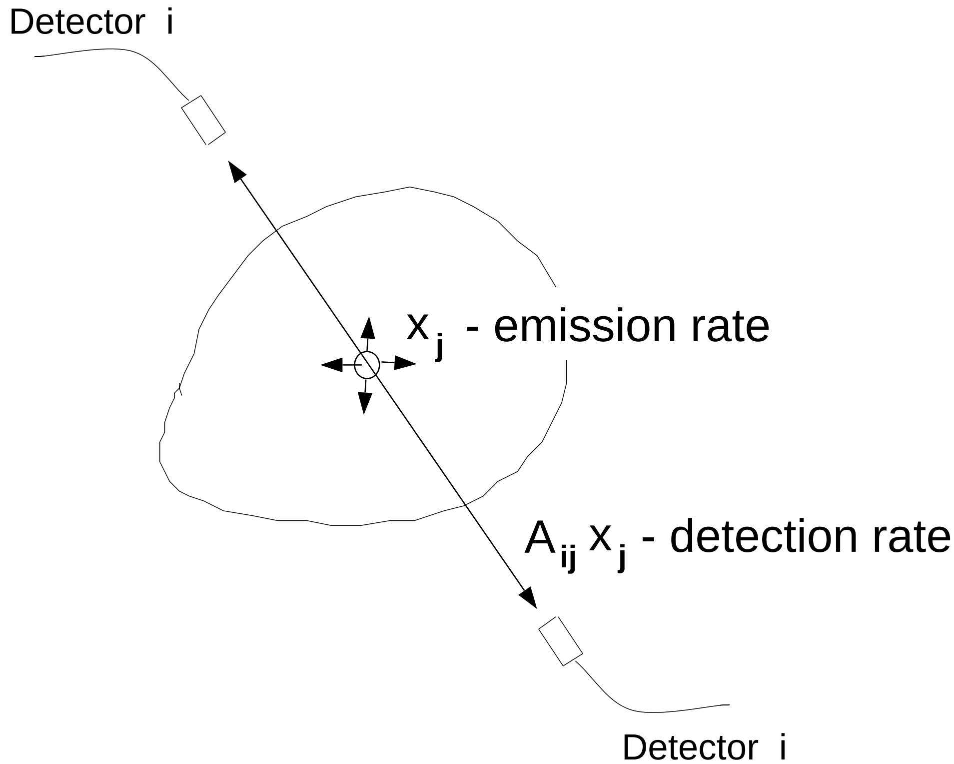
$$\int_0^T \mu(t)dt \cong -\log\left(\frac{Y_T}{\lambda_0}\right)$$

where:

$\lambda_0$ is the dosage

$Y_T$ is the photon count at the detector

# Positron Emission Tomography (PET)

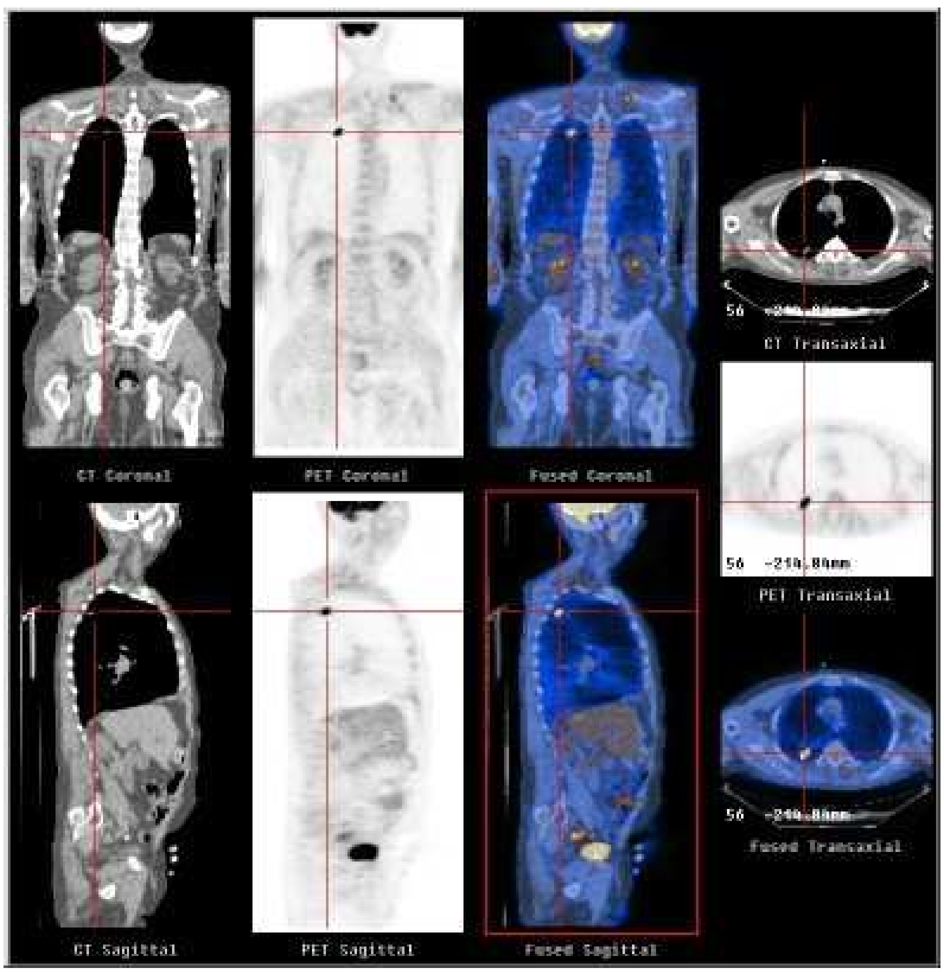



$$E[y_i] = \sum_j A_{ij} x_j$$

- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements

# Example: PET/CT Scan

- Generally low space/time resolution
- Little anatomical detail ⇒ couple with CT
- Can detect disease



# Coordinate Rotation

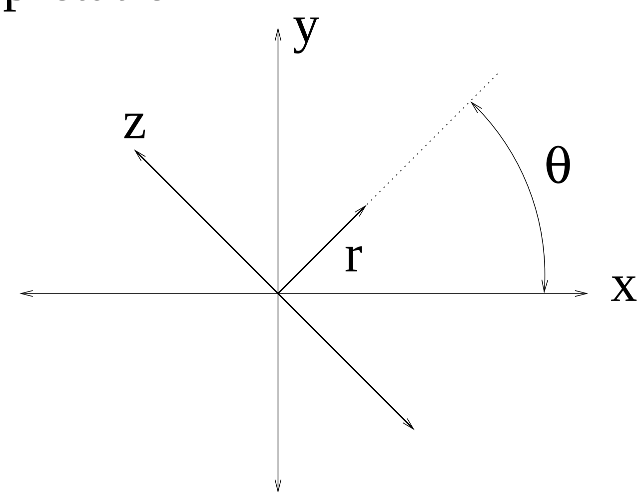
- Define the counter-clockwise rotation matrix

$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Define the new coordinate system $(r, z)$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$$

- Geometric interpretation

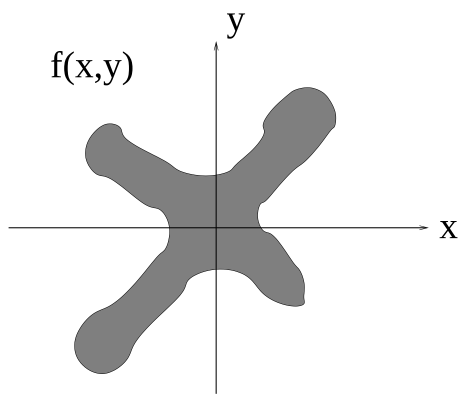



- Inverse transformation

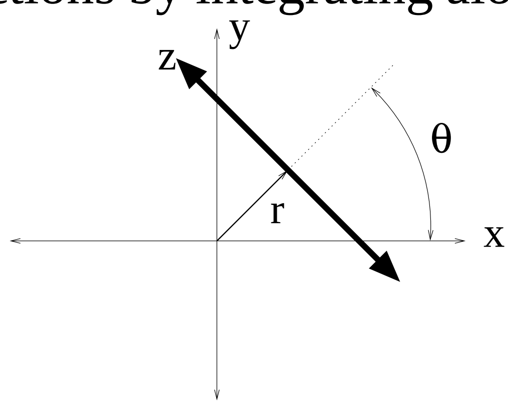
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

# Integration Along Projections

- Consider the function $f(x, y)$.




- We compute projections by integrating along $z$ for each $r$.




- The projection integral for each $r$ and $\theta$ is given by

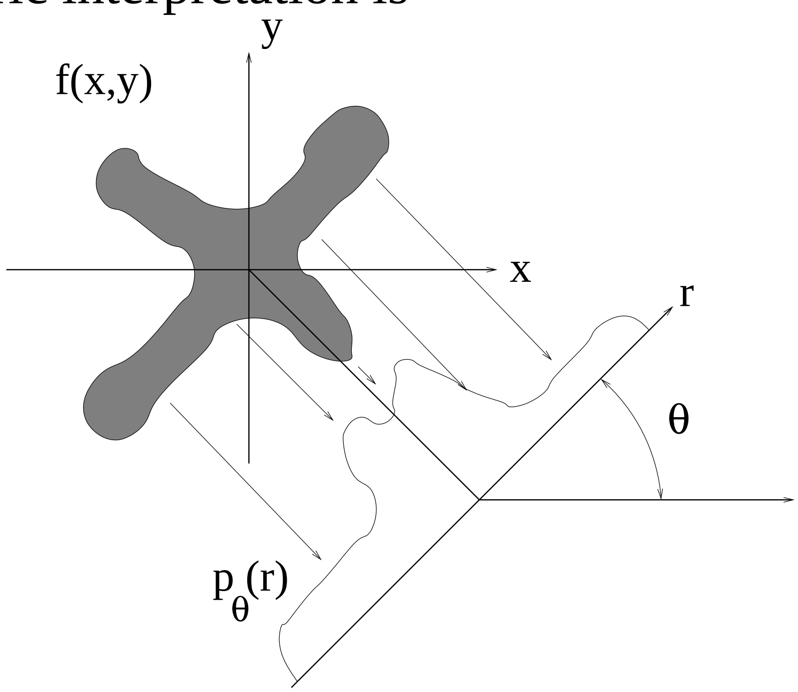
$$
\begin{aligned}
p_\theta(r) &= \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz
\end{aligned}
$$

# The Radon Transform

- The Radon transform of the function $f(x, y)$ is defined as

  $$p_\theta(r) = \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz$$

- The geometric interpretation is




Notice that the projection corresponding to $r = 0$ goes through the point $(x, y) = (0, 0)$.

# The Fourier Slice Theorem

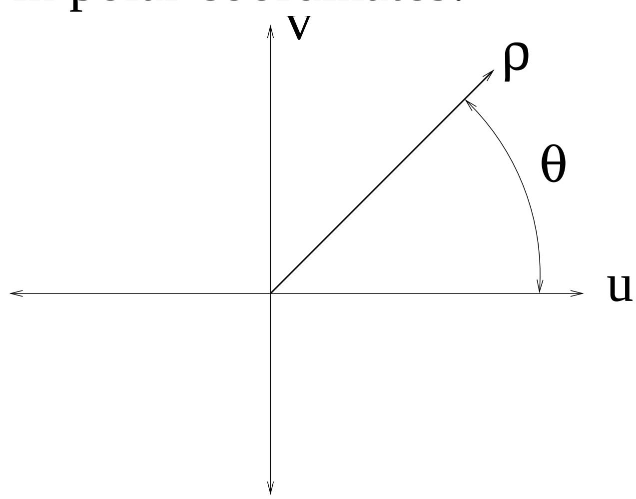
- Let

$$\begin{aligned} P_\theta(\rho) &= CTFT\left\{p_\theta(r)\right\} \\ F(u,v) &= CSFT\left\{f(x,y)\right\} \end{aligned}$$

  Then

$$P_\theta(\rho) = F(\rho\cos(\theta), \rho\sin(\theta))$$

- $P_\theta(\rho)$ is $F(u,v)$ in polar coordinates!



# Proof of the Fourier Slice Theorem

- By definition

$$p_\theta(r) \ = \ \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz$$

- The CTFT of $p_\theta(r)$ is then given by

$$\begin{aligned} P_\theta(\rho) \ &= \ \int_{-\infty}^{\infty} p_\theta(r) e^{-j2\pi\rho r} dr \\ &= \ \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz\right] e^{-j2\pi\rho r} dr \\ &= \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) e^{-j2\pi\rho r} dz dr \end{aligned}$$

- We next make the change of variables

$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} \ .$$

Notice that the Jacobian is $|\mathbf{A}_\theta| = 1$, and that $r = x\cos(\theta) + y\sin(\theta)$. This results in

$$\begin{aligned} P_\theta(\rho) \ &= \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\,(x,y)\, e^{-j2\pi\rho[x\cos(\theta)+y\sin(\theta)]} dx dy \\ &= \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\,(x,y)\, e^{-j2\pi[x\rho\cos(\theta)+y\rho\sin(\theta)]} dx dy \\ &= \ F(\rho\cos(\theta), \rho\sin(\theta)) \end{aligned}$$

# Alternative Proof of the Fourier Slice Theorem

- First let $\theta = 0$, then

$$p_0(r) = \int_{-\infty}^{\infty} f(r,y)\, dy$$

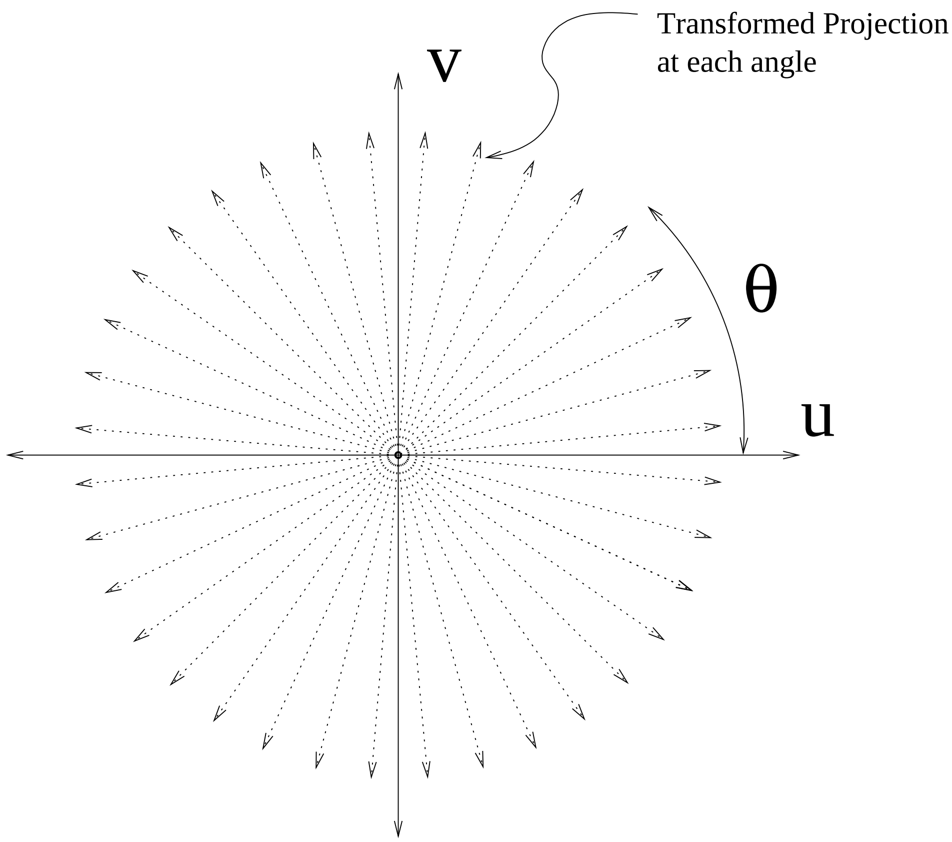
Then

$$\begin{aligned} P_0(\rho) &= \int_{-\infty}^{\infty} p_0(r) e^{-2\pi j r\rho}\, dr \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(r,y)\, dy\right] e^{-2\pi j r\rho}\, dr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,y) e^{-2\pi j (r\rho + y0)}\, dr\, dy \\ &= F(\rho, 0) \end{aligned}$$

- By rotation property of CSFT, it must hold for any $\theta$.

# Inverse Radon Transform

- Physical systems measure $p_\theta(r)$.
- From these, we compute $P_\theta(\rho) = CTFT\{p_\theta(r)\}$.




- Next we take an inverse CSFT to form $f(x, y)$.

  **Problem:** This requires polar to rectagular conversion.

  **Solution:** Convolution backprojection

# Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of $F(u,v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du\,dv = |\rho| d\theta\, d\rho$$

- This results in the expression

$$\begin{aligned} f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{2\pi j(xu+yv)}\, dudv \\ &= \int_{-\infty}^{\infty}\int_{0}^{\pi} P_\theta(\rho) e^{2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))}\, |\rho| d\theta\, d\rho \\ &= \int_0^\pi \underbrace{\left[\int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho(x\cos(\theta)+y\sin(\theta))}\, d\rho\right]}_{g_\theta(x\cos(\theta)+y\sin(\theta))} d\theta \end{aligned}$$

- Then $g(t)$ is given by

$$\begin{aligned} g_\theta(t) &= \int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho t}\, d\rho \\ &= CTFT^{-1}\left\{|\rho| P_\theta(\rho)\right\} \\ &= h(t) * p_\theta(r) \end{aligned}$$

where $h(t) = CTFT^{-1}\{|\rho|\}$, and

$$f(x,y) = \int_0^\pi g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

# Summary of CBP Algorithm

1. Measure projections $p_\theta(r)$.
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.
3. Back project filtered projections

$$f(x, y) = \int_0^\pi g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

# A Closer Look at Projection Filter

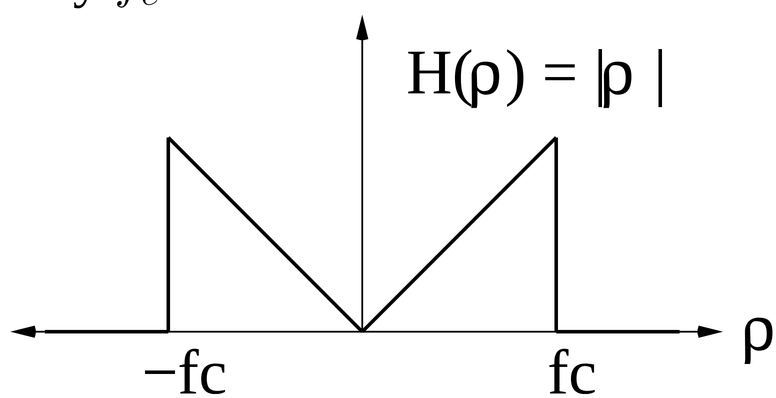
1. At each angle, projections are filtered.

$$g_\theta(r) = h(r) * p_\theta(r)$$

2. The frequency response of the filter is given by

$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to $|\rho| \leq f_c$ for some cut-off frequency $f_c$.




So

$$\begin{aligned} H(\rho) &= f_c \left[\text{rect}\left(f/(2f_c)\right) - \Lambda\left(f/f_c\right)\right] \\ h(r) &= f_c^2 \left[2\text{sinc}(t2f_c) - \text{sinc}^2(tf_c)\right] \end{aligned}$$

# A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^\pi b_\theta (x, y)\, d\theta$$

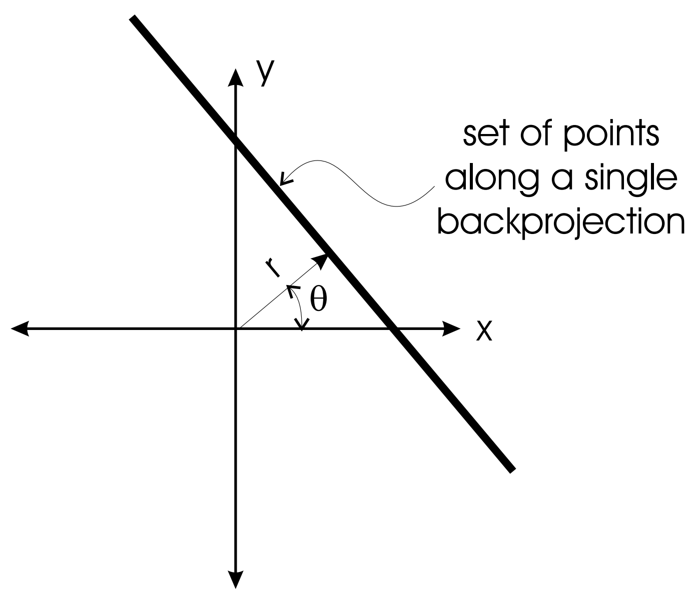
where

$$b_\theta(x, y) = g_\theta(x \cos(\theta) + y \sin(\theta))$$

- Consider the set of points $(x, y)$ such that

$$r = x \cos(\theta) + y \sin(\theta)$$

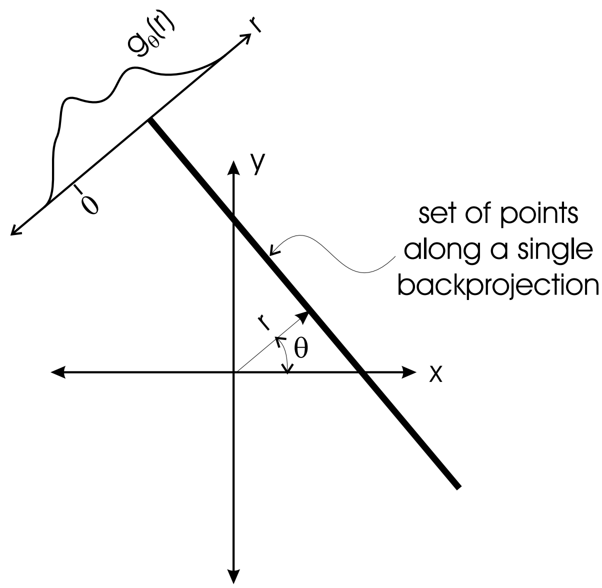
This set looks like




- Along this line $b_\theta(x, y) = g_\theta(r)$.

# Back Projection Continued

- For each angle $\theta$ back projection is constant along lines of angle $\theta$ and takes on value $g_\theta(r)$.




- Complete back projection is formed by integrating (summing) back projections for angles ranging from $0$ to $\pi$.

$$
\begin{aligned}
f(x,y) &= \int_0^\pi b_\theta(x,y)\, d\theta \\
&\approx \frac{\pi}{M} \sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x,y)
\end{aligned}
$$

- Back projection "smears" values of $g(r)$ back over image, and then adds smeared images for each angle.