EE 637 Final

May 1, Spring 2006

Name: Key

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 40 points for a total score of 200 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator.

Good Luck.

Passive acceptance of the teachers wisdom is easy to most boys and girls. It involves no effort of independent thought, and seems rational because the teacher knows more than his pupils; it is moreover the way to win the favor of the teacher unless he is a very exceptional man. Yet the habit of passive acceptance is a disastrous one in later life. It causes man to seek and to accept a leader, and to accept as a leader whoever is established in that position. Bertrand Russel

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(40pt)

Let $T(m,n)$ be a 2-D random field of i.i.d. random variables which are uniformly distributed on the interval $[0,1]$. These thresholds are used to produce a binary halftone $B(m,n)$ for the gray level g using the relationship

$0 \leq g \leq 1$

$$B(m,n) = \begin{cases} 1 & \text{if } g \geq T(m,n) \\ 0 & \text{if } g < T(m,n) \end{cases}$$

a) Is $B(m,n)$ a stationary random process?

b) Calculate μ, the mean of $B(m,n)$.

c) Calculate σ^2, the variance of $B(m,n)$.

d) Calculate $R(k,l) = E\left[D(m,n)D(m+k,n+l)\right]$ were $D(m,n) = B(m,n) - \mu$.

e) Calculate the power spectral density $S(e^{j\mu}, e^{j\nu})$ of $D(m,n)$.

f) Will $B(m,n)$ be a good quality halftone for the gray level g? Justify your answer.

g) Which gray levels will look most noisy? Which will look least noisy?

a) yes

b) $\mu = E[B(m,n)] = P\{g \geq T(m,n)\}$

$= g$

c) $\sigma^2 = E\left[(B(m,n) - \mu)^2\right]$

$= E\left[B^2(m,n)\right] - \mu^2$

$= P\{g \geq T(m,n)\} - g^2$

$= g - g^2 = g(1-g)$

d) $R(k,l) = E\left[D(m,n)D(m+k,n+l)\right]$

$= g(1-g)\,\delta(k,l)$

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e) $S(e\vartheta\mu, e\vartheta\nu) = g(1-g)$

f) No Too much high frequency noise.

g) $g = 1/2 \Rightarrow$ most noisy

$g = 0, 1 \Rightarrow$ least noisy

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Problem 2.(40pt)

Consider the color space given by

$$\begin{bmatrix} r \\ g \\ b \end{bmatrix} = M \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where X, Y, and Z denote the standard CIE color space, and $[r, g, b]$ are the tristimulus values corresponding to physically displayed red (R), green (G), and blue (B) color primaries.

a) Are the entries in the matrix M positive (i.e. ≥ 0), negative (i.e. < 0) or a combination of positive and negative. Justify your answer.

b) Are the entries in the matrix M^{-1} positive (i.e. ≥ 0), negative (i.e. < 0) or a combination of positive and negative. Justify your answer.

c) What does the second column of M^{-1} represent?

d) What does it tell you if the rows of M^{-1} sum to 1?

a) Both positive and negative because X, Y, Z correspond to imaginary primaries which are the columns of M.

b) The entries in M^{-1} are all positive because r, g, b primaries can always be specified with positive X Y Z values.

c) The green primary in X, Y, Z coordinates

d) The white point is equal energy white.

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Problem 3.(40pt)

Let $\{X_i\}_{i=1}^N$ and $\{Y_i\}_{i=1}^N$ be i.i.d. Gaussian random variables with mean μ and variance 1. We will refer to X_i as training data, and Y_i as testing data.

Let

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

be an estimate of μ, and let

$$\begin{aligned}\gamma_X(N) &= \frac{1}{N}\sum_{i=1}^{N}(X_i - \hat{\mu})^2 \\ \gamma_Y(N) &= \frac{1}{N}\sum_{i=1}^{N}(Y_i - \hat{\mu})^2\end{aligned}$$

be the prediction variance for the training and testing data respectively.

It is well known that

$$E[\gamma_X(N)] = \frac{N-1}{N}$$

a) Calculate $E[\gamma_Y(N)]$ as a function of N.

b) Sketch the $E[\gamma_Y(N)]$ and $E[\gamma_X(N)]$ as a function of N.

c) Explain the behavior of these plots as a function of N, and relate it to what you know about the use of training and testing data.

d) You attend a conference and hear a paper about compressing images using least squares linear predication. The author used the "Lenna" image to train a prediction filter with a 100×100 prediction window, and the decompressed version of the "Lenna" image looks great compared to JPEG compression at a similar bit rate. Critique this work.

a) $E[\gamma_Y(N)] = E\left[\frac{1}{N}\sum_{i=1}^{N}(Y_i-\hat{\mu})^2\right]$

$= E\left[\frac{1}{N}\sum_{i=1}^{N}(Y_i-\mu)^2\right] + E\left[\frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}-\mu)^2\right]$

$+ E\left[\frac{1}{N}\sum_{i=1}^{N}(Y_i-\mu)(\hat{\mu}-\mu)\right]$

$= 1 + \frac{1}{N} + 0 = \frac{N+1}{N}$

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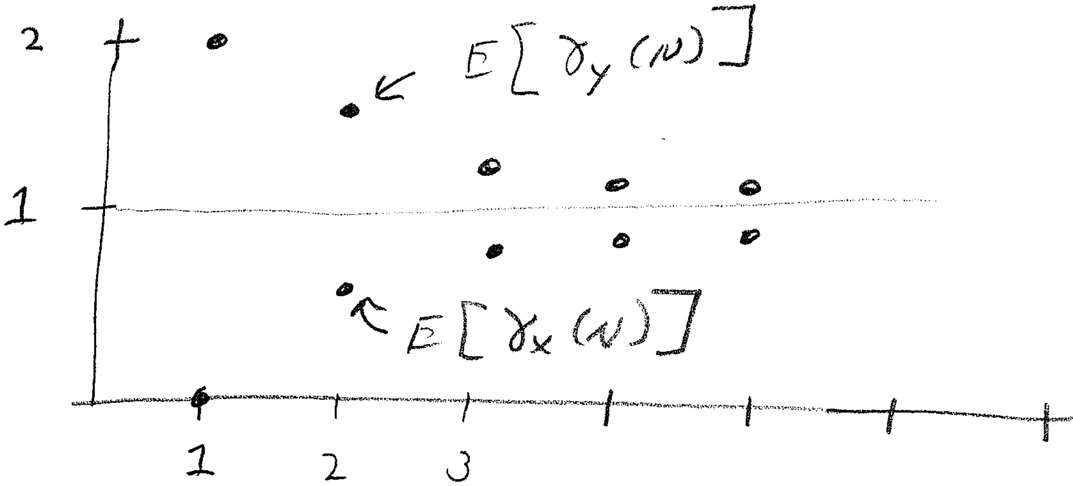
b)



c) $E[\gamma_x(N)]$ ← Training data is an increasing function of N

$E[\gamma_y(N)]$ ← testing data is a decreasing function of N

$$\lim_{N\to\infty} E[\gamma_x(N)] = \lim_{N\to\infty} E[\gamma_y(N)]$$

d) There may be a problem because the author tested on training data, and trained with a large number of parameters (i.e. 100^2)

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Problem 4.(40pt)

Consider the set of data $\{x_n\}_{n=0}^{N-1}$ for N odd. We would like to estimate a "central value" using a method known as M-estimation. To do this we compute the following function

$$\hat{\theta} = \arg\min_{\theta} \left\{ \sum_{n=0}^{N-1} \rho(x_n - \theta) \right\}$$

where ρ is a function with the properties that $\rho(\Delta) \geq 0$ and $\rho(-\Delta) = \rho(\Delta)$.

a) What function $\rho(\Delta)$ will result in the mean as shown below?

$$\text{mean is } \hat{\theta} = \frac{1}{N} \sum_{i=0}^{N-1} x_i$$

b) What function $\rho(\Delta)$ will result in the median?

c) Select a function $\rho(\Delta)$ which usually produces an estimate close to the mean, but limits the influence of a single value of x_i.

a) $\rho(\Delta) = |\Delta|^2$

b) $\rho(\Delta) = |\Delta|$

c) $\rho(\Delta) = \begin{cases} |\Delta|^2 & \text{for } |\Delta| < T \\ |2T\Delta| - T^2 & \text{for } |\Delta| > T \end{cases}$

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Problem 5.(40pt)

Consider an image $f(x,y)$ with a CSFT given by $F(u,v)$ and a forward projection

$$\begin{aligned} p_\theta(r) &= \mathcal{FP}\{f(x,y)\} \\ &= \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta))\, dz \ . \end{aligned}$$

The image is reconstructed by first filtering the projections with the filter

$$H(\rho) = \begin{cases} |\rho| & \text{for } |\rho| < f_c \\ 0 & \text{for } |\rho| \geq f_c \end{cases}$$

(where ρ is the frequency domain variable corresponding to r) and then backprojecting using the equation

$$\tilde{f}(x,y) = \int_0^{\theta_c} g_\theta(x\cos(\theta) + y\sin(\theta))d\theta \ .$$

In class, we choose $\theta_c = \pi$ to reconstruct the image.

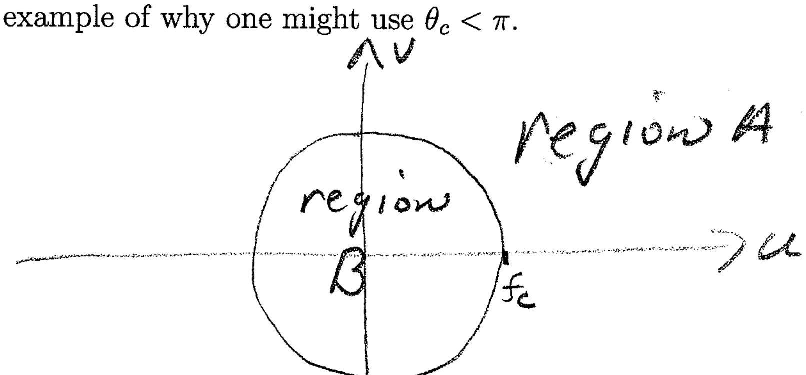
a) When $f_c < \infty$ and $\theta_c = \pi$, will $f(x,y)$ and $\tilde{f}(x,y)$ always be equal for this case? When will they be equal? When will they not be equal? (Hint: You may want to use a sketch of $F(u,v)$.)

b) When $f_c = \infty$ and $\theta_c = \pi/2$, will $f(x,y)$ and $\tilde{f}(x,y)$ always be equal for this case? When will they be equal? When will they not be equal? (Hint: You may want to use a sketch of $F(u,v)$.)

c) What are the advantages and disadvantages of using a large value for f_c?

d) Give an example of why one might use $\theta_c < \pi$.

a)



If $F(u,v)$ is only contained in region B $\Rightarrow f(x,y) = \tilde{f}(x,y)$

If $F(u,v)$ is has energy in region A $\Rightarrow f(x,y) \neq \tilde{f}(x,y)$

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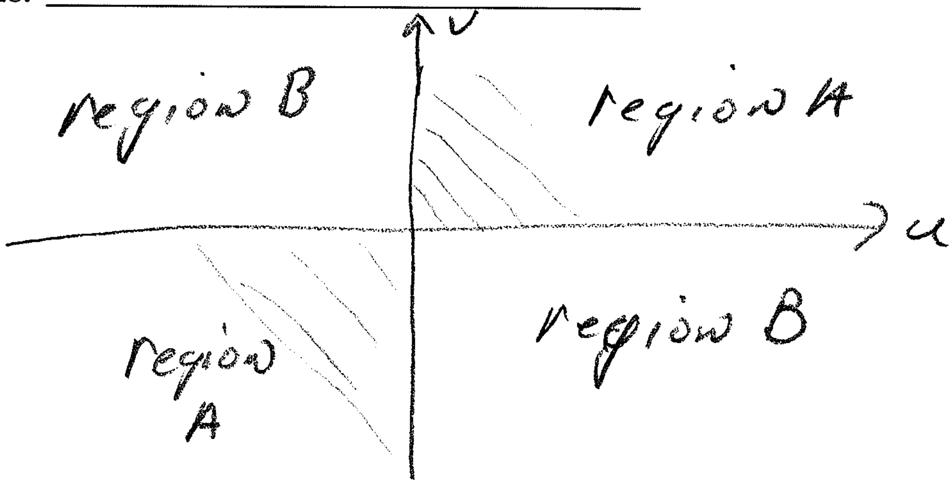
b)



If $F(u,v)$ is only contained in region A

$\Rightarrow f(x,y) = \tilde{f}(x,y)$

If $F(u,v)$ has energy in region B

$\Rightarrow f(x,y) \neq \tilde{f}(x,y)$

c) Advantage $\Rightarrow$ higher resolution

Disadvantage $\Rightarrow$ more noise

d) If projection angles are limited by dense occluding object.