EE 637 Final Exam

May 5, Spring 2005

Name: ______________________________________

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each part of each problem is worth 10 points for a total score of 200 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}} [X(f)]$$

$$\mathrm{rep}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(40pt)

Consider a lossless predictive coder which predicts the pixel $X_{s_1,s_2} = k$ from the two pixels $X_{s_1,s_2-1} = i$ and $X_{s_1-1,s_2} = j$. In order to design the predictor, you first measure the histogram for the values of i, j, k from some sample images. This results in the following measurements.

i	j	k	$h(i,j,k)$
0	0	0	30
0	0	1	2
0	1	0	4
0	1	1	12
1	0	0	12
1	0	1	4
1	1	0	2
1	1	1	30

a) Use the values of $h(i,j,k)$ to calculate $\hat{p}(k|i,j)$, an estimate of

$$p(k|i,j) = P\left\{X_{s_1,s_2} = k | X_{s_1,s_2-1} = i,\ X_{s_1-1,s_2} = j\right\}$$

and use them to fill in the table below.

i	j	k	$\hat{p}(k|i,j)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

b) For each value of i, and j, compute a binary valued estimate of X_{s_1,s_2} and use it to fill in the table below.

i	j	$\hat{X}_{s_1,s_2}$
0	0	
0	1	
1	0	
1	1	

c) Draw a block diagram for the lossless predictive coder. The block diagram should include an entropy coder.

d) Assuming the prediction errors are independent (but not identically distributed), calculate an expression for the theoretically achievable bit rate for the lossless predictive encoder. Justify your answer. (Hint: Use the fact that $\log_2(3) = 1.585$, $\log_2(7) = 2.807$, and $\log_2(15) = 3.907$.)

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Problem 2.(40pt)

Below is a partial pseudo-code description of a subroutine called *ConnectedSet* for labeling all pixels connected to the pixel s_0. Let $c(s)$ be the set of connected neighbors to the pixel s, and let Y be the image containing the label for each pixel.

Initialize $Y_r = 0$ for all $r \in S$
$ClassLabel = 1$
ConnectedSet$(s_0, Y, ClassLabel)$ {
    $B \leftarrow \{s_0\}$
    While B is not empty {
        $s \leftarrow$ any element of B

        ⟨**Missing Pseudo-Code**⟩

    }
    return(Y)
}

a) Fill in the missing section of code with the correct pseudo-code operations.

For the following problems, let $c(s) = \{r \in \partial s : X_r = X_s\}$ where ∂s is the set of neighbors for the pixel s, and the image X has the following specific form.

The image X

		j				
		0	1	2	3	4
i	0	1	0	0	0	0
	1	1	0	0	0	0
	2	0	1	1	0	0
	3	0	1	1	0	0
	4	0	1	0	0	1

b) Use the following table to specify the values of Y returned by the subroutine when $s_0 = (i, j) = (0, 0)$ and a 4-point neighborhood is used.

c) Use the following table to specify the values of Y returned by the subroutine when $s_0 = (i, j) = (0, 0)$ and a 8-point neighborhood is used.

d) Use the following table to specify the values of Y returned by the subroutine when $s_0 = (i, j) = (0, 4)$ and a 8-point neighborhood is used.

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Problem 3.(40pt)

Consider the 1-D error diffusion filter with input x_n, binary output $b_n \in \{0, 1\}$, modified input $\tilde{x}_n$, quantizer output $Q(\tilde{x}_n)$, and filter $h_n = \delta_{n-1}$.

a) Draw a block diagram for the 1-D error diffusion filter, and write out the equations for this system.

b) Assuming that $x_n = 0.25$ and the quantizer has the form

$$Q(\tilde{x}) = \begin{cases} 1 & \text{if } \tilde{x} \geq 0.5 \\ 0 & \text{if } \tilde{x} < 0.5 \end{cases}$$

calculate the output b_n for $n = 0, \cdots, 6$. (Assume that the error diffusion filter starts with an error of 0 for all $n < 0$.

For the next questions, assume that the quantizer has M output levels and step sizes of $\frac{1}{M}$ where M is large.

c) What is a good assumption for the power spectrum of the quantizer errors $z_n = \tilde{x}_n - Q(\tilde{x}_n)$ when M is large?

d) Compute the approximate power spectrum of the display error $b_n - x_n$ when M is large.

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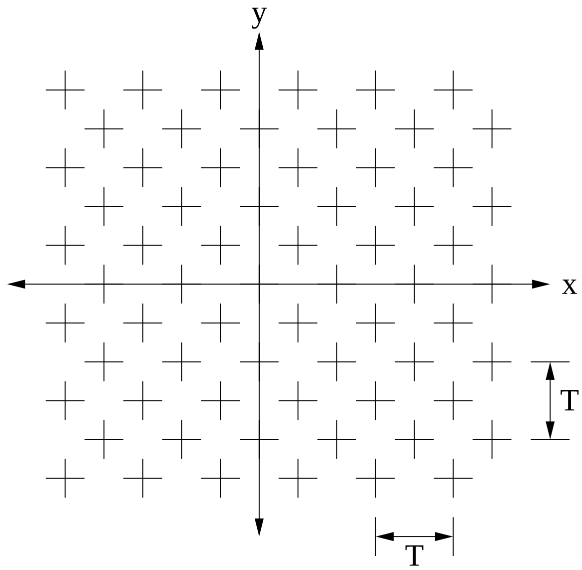
Problem 4.(40pt)

In this problem, you will anaylze the effect of a diagonal grid sampling pattern. Consider a 2-D function $f(x, y) = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ where $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ is a continuously valued vector. For a particular application, $f(x, y)$ is sampled on the non-rectangular grid shown below where the crosses indicate the location of a sample and $T = \sqrt{2}$.



Also define the function

$$\tilde{f}(x, y) = \tilde{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(A \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

where A is a 2×2 matrix.

a) Find an orthonormal matrix A so that the sampling of $\tilde{f}$ is on a rectangular grid with sampling period 1.

For the following problems, let $F(u, v)$ be the CSFT of $f(x, y)$ and let $\tilde{F}(u, v)$ be the CSFT of $\tilde{f}(x, y)$.

b) Assuming that $\tilde{F}(u, v)$ is band limited to $|u| < f_c$ and $|v| < f_c$, what is the maximum value of f_c which guarantees that no aliasing occurs? Sketch the region of support for $\tilde{F}(u, v)$. (i.e. the region in the (u, v) plane such that $\tilde{F}(u, v)$ is not necessarily zero.)

c) Sketch the region of support for $F(u, v)$. (i.e. the region in the (u, v) plane such that $F(u, v)$ is not necessarily zero.)

d) What is the maximum horizontal frequency of $f(x, y)$ that can be reconstructed using the diagonal grid sampling pattern? What is the maximum horizontal frequency of $\tilde{f}(x, y)$ that can be reconstructed using the rectangular grid sampling pattern?

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Problem 5.(40pt)

Let $X(m,n) = g$ be an image taking on values in the range 0 to 255 that is displayed on an 8-bit gamma corrected display with a gamma value of γ and a maximum luminance of I_o. Let

$$B(m,n) = 255\left(1 + (-1)^m(-1)^n\right)/2$$

be an image formed by a checker-board pattern of 0's and 255's that is displayed on the same display with a gamma value of γ and a maximum luminance of I_o.

a) What is the average luminance of the displayed image $B(m,n)$?

b) What is the average luminance of the displayed image $X(m,n)$ in terms of the value g?

In the following questions, assume that the value of g is chosen so that the average luminance of the two images match.

c) Write an equation in which relates of g and γ for this case.

d) If $g = \frac{255}{\sqrt{2}}$, than what is the value of γ?