# **Nonlinear Filtering**

#### • Linear filters

- Tend to blur edges and other image detail.
- Perform poorly with non-Gaussian noise.
- Result from Gaussian image and noise assumptions.
- Images are not Gaussian.

#### • Nonlinear filter

- Can preserve edges
- Very effective at removing impulsive noise
- Result from non-Gaussian image and noise assumptions.
- Can be difficult to design.

## **Linear Filters**

• Definition: A system y=T[x] is said to be linear if for all  $\alpha,\beta\in I\!\!R$ 

$$\alpha y_1 + \beta y_2 = T[\alpha x_1 + \beta x_2]$$

where  $y_1 = T[x_1]$  and  $y_2 = T[x_2]$ .

• Any filter of the form

$$y_s = \sum_r h_{s,r} x_r$$

# **Homogeneous Filters**

• Definition: A filter y=T[x] is said to be homogeneous if for all  $\alpha\in I\!\!R$ 

$$\alpha y = T[\alpha x]$$

- This is much weaker than linearity.
- Homogeneity is a natural condition for scale invariant systems.

## **Median Filter**

- ullet Let W be a window with an odd number of points.
- Then the median filter is given by

$$y_s = \text{median} \{x_{s+r} : r \in W\}$$

- Is the median filter:
  - Linear?
  - Homogeneous?
- Consider the 1-D median filter with a 3-point window.

# Median Filter: Optimization Viewpoint

• Consider the median filter

$$y_s = \text{median} \{x_{s+r} : r \in W\}$$

and consider the following functional.

$$F(\theta) \stackrel{\triangle}{=} \sum_{r \in W} |\theta - x_{s+r}|$$

ullet Then  $y_s$  solves the following optimization equation.

$$y_s = \arg\min_{\theta} F(\theta)$$

• Differentiating, we have

$$\frac{d}{d\theta}F(\theta) = \frac{d}{d\theta} \sum_{r \in W} |\theta - x_{s+r}|$$

$$= \sum_{r \in W} \operatorname{sign}(\theta - x_{s+r})$$

$$\stackrel{\triangle}{=} f(\theta)$$

This expression only holds for  $\theta \neq x_{s+r}$  for all  $r \in W$ .

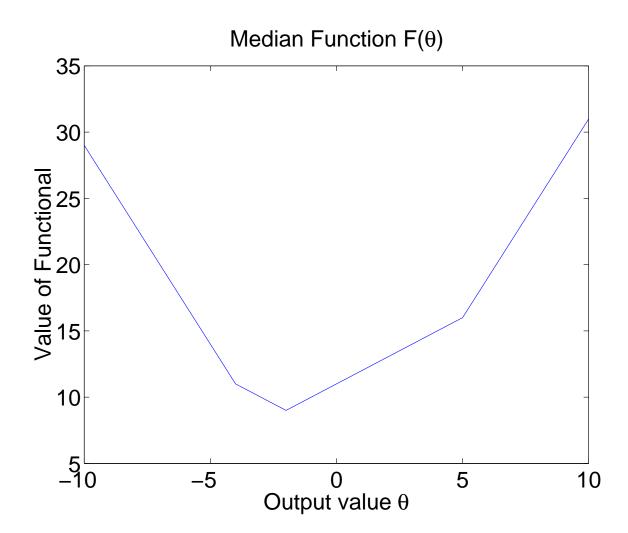
ullet So the solution falls at  $\theta=x_{s*}$  such that

$$0 = \sum_{\substack{r \in W \\ r \neq (s*-s)}} \operatorname{sign}(\theta - x_{s+r})$$

# **Example: Median Filter Function**

- Consider a 1-D median filter
  - Three point window of  $W = \{-1, 0, 1\}$
  - Inputs [x(n-1), x(n), x(n+1)] = [-2, -4, 5].

$$F(\theta) = \sum_{k=-1}^{1} |\theta - x_{n+k}|$$

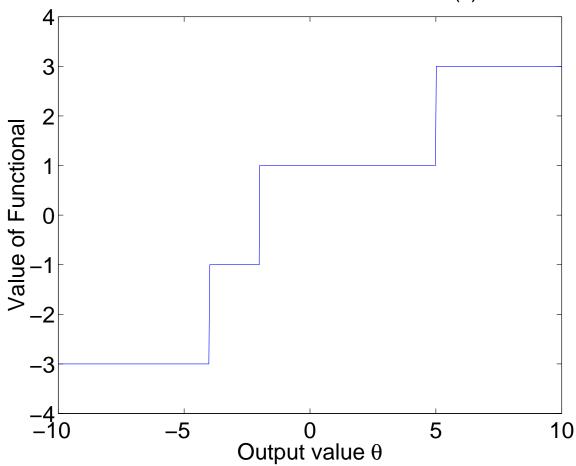


# **Example: Derivative of Median Filter Function**

- Consider a 1-D median filter
  - Three point window of  $W = \{-1, 0, 1\}$
  - Inputs [x(n-1), x(n), x(n+1)] = [-2, -4, 5].

$$f(\theta) = \sum_{k=-1}^{1} \operatorname{sign}(\theta - x_{n+k})$$

### Derivative of Median Function $f(\theta)$

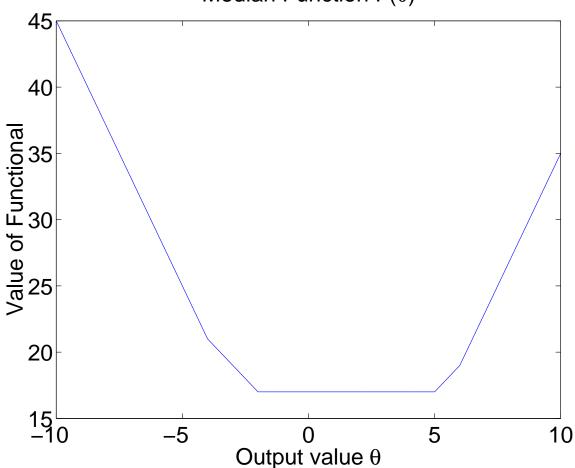


## **Problem with an Even Number of Points**

- Consider a 1-D median filter
  - Four point window of  $W = \{-1, 0, 1, 2\}$
  - Inputs [x(n-1), x(n), x(n+1), x(n+2)] = [-2, -4, 5, 6].
- Solution is not unique.

$$F(\theta) = \sum_{k=-1}^{2} |\theta - x_{n+k}|$$

## Median Function $F(\theta)$



## Weighted Median Filter

Defined the functional

$$F(\theta) \stackrel{\triangle}{=} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

where  $a_r$  are weights assigned to each point in the window W.

Weighted median is computed by

$$y_s = \arg\min_{\theta} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

• Differentiating, we have

$$\frac{d}{d\theta}F(\theta) = \frac{d}{d\theta} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

$$= \sum_{r \in W} a_r \operatorname{sign}(\theta - x_{s+r})$$

$$\stackrel{\triangle}{=} f(\theta)$$

This expression only holds for  $\theta \neq x_r$  for all  $r \in W$ .

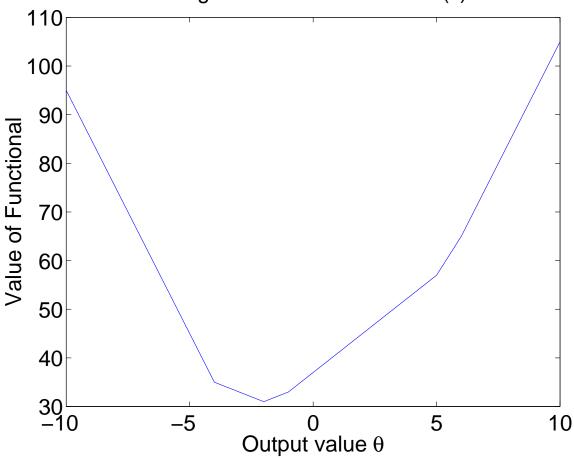
• Need to find s\* such that  $f(\theta)$  is "nearly" zero.

## **Example: Weighted Median Filter Function**

- Consider a 1-D median filter
  - Five point window of  $W = \{-2, -1, 0, 1, 2\}$
  - Inputs  $[x(n-2), \dots, x(n+2)] = [6, -2, -4, 5, -1].$
  - Weights [a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1].

$$F(\theta) = \sum_{k=-1}^{1} a(k) |\theta - x_{n+k}|$$

## Weighted Median Function $F(\theta)$

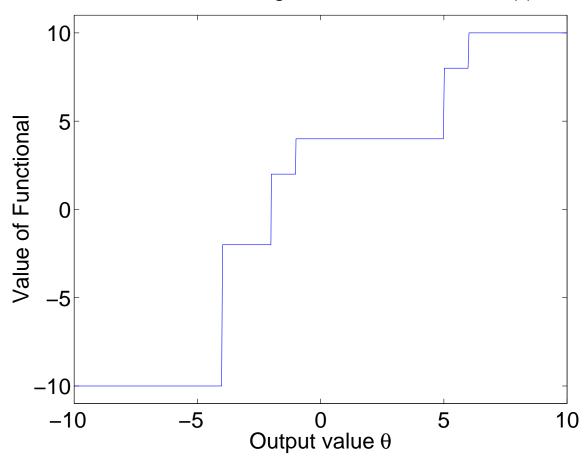


# **Example: Derivative of Median Filter Function**

- Consider a 1-D median filter
  - Five point window of  $W = \{-2, -1, 0, 1, 2\}$
  - Inputs  $[x(n-2), \dots, x(n+2)] = [6, -2, -4, 5, -1].$
  - Weights [a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1].

$$f(\theta) = \sum_{k=-1}^{1} a(k) \operatorname{sign}(\theta - x_{n+k})$$

### Derivative of Weighted Median Function $f(\theta)$



# **Computation of Weighted Median**

- 1. Sort points in window.
  - Let  $x_{(1)} < x_{(2)} < \cdots < x_{(p)}$  be the sorted values.
  - These values are known as order statistics.
  - Let  $a_{(1)}, a_{(2)}, \dots, a_{(p)}$  be the **corresponding** weights.
- 2. Find i\* such that the following equations hold

$$a_{i*} + \sum_{i=1}^{i*-1} a_{(i)} \ge \sum_{i=i*-1}^{p} a_{(i)}$$

$$\sum_{i=1}^{i*-1} a_{(i)} \le \sum_{i=i*-1}^{p} a_{(i)} + a_{i*}$$

3. Then the value  $a_{(i*)}$  is the weighted median value.

# **Comments on Weighted Median Filter**

- Weights may be adjusted to yield the "best" filter.
- Largest and smallest values are ignored.
- Same as median filter for  $a_r = 1$ .