

## Filtered Random Processes

- Consider the 2-D linear system

$$Y(m, n) = h(m, n) * X(m, n)$$

where  $X(m, n)$  is a 2-D wide sense stationary random process.

- It may be easily shown that

$$R_y(m, n) = h(m, n) * h(-m, -n) * R_x(m, n)$$

$$S_y(e^{j\mu}, e^{j\nu}) = |H(e^{j\mu}, e^{j\nu})|^2 S_x(e^{j\mu}, e^{j\nu})$$

## White Noise

- Definition:
  - $X(m, n)$  - independent identically distributed (i.i.d.) Gaussian random variables with distribution  $N(0, 1)$ .
- Then
  - $X(m, n)$  is wide sense stationary with

$$\mu(m, n) = 0$$

$$\begin{aligned} R_x(k, l) &= E[X(0, 0)X(k, l)] \\ &= \delta(k, l) \end{aligned}$$

$$\begin{aligned} S_x(e^{j\mu}, e^{j\nu}) &= DSFT \{R_x(k, l)\} \\ &= 1 \end{aligned}$$

## Filtered White Noise

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- $Y(m, n) = h(m, n) * X(m, n)$ .

- Then

- $Y(m, n)$  is wide sense stationary with

$$\begin{aligned} S_y(e^{j\mu}, e^{j\nu}) &= |H(e^{j\mu}, e^{j\nu})|^2 S_x(e^{j\mu}, e^{j\nu}) \\ &= |H(e^{j\mu}, e^{j\nu})|^2 \cdot 1 \end{aligned}$$

$$R_y(k, l) = h(m, n) * h(-m, -n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m, n) h(m + k, n + l)$$

- $R_y(k, l)$  is the autocorrelation of  $h(m, n)$  with itself.

## Causal Prediction

- $Y_s$  is a 2-D wide sense stationary Gaussian random process.
- Define
  - The past values are  $Y_{<s} = \{Y_r : r < s\}$ .
  - The minimum mean squared error (MMSE) predictor of  $Y_s$  given the past is

$$\hat{Y}_s = E[Y_s | Y_{<s}]$$

- The prediction error is  $X_s = Y_s - \hat{Y}_s$ .

## Properties of Causal Predictors

- Fact 1: (WLOG, assume  $r < s$ .)

$$\begin{aligned}
 E[X_s X_r] &= E[E[X_s X_r | Y_{<s}]] \\
 &= E[E[(Y_s - \hat{Y}_s)(Y_r - \hat{Y}_r) | Y_{<s}]] \\
 &= E[E[(Y_s - \hat{Y}_s) | Y_{<s}](Y_r - \hat{Y}_r)] \\
 &= E[(E[Y_s | Y_{<s}] - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
 &= E[(\hat{Y}_s - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
 &= E[0(Y_r - \hat{Y}_r)] = 0
 \end{aligned}$$

- Fact 2:  $\sigma^2 \triangleq E[X_s^2]$  is the prediction variance.
- Fact 3: The predictor must be space invariant and linear.

$$\hat{Y}_s = \sum_{r > (0,0)} h_r Y_{s-r}$$

## Autoregressive (AR) Processes

- Definitions:
  - $Y_s$  - 2-D wide sense stationary Gaussian random process.
  - $h_s$  - MMSE linear predictor for  $Y_s$ .
  - $X_s = Y_s - h_s * Y_s$  - predictor error.
- If  $h_s$  is FIR, then  $Y_s$  is known as an autoregressive (AR) process.

## Properties of AR Processes

- Remember that

$$X_s = Y_s - h_s * Y_s$$

- Then

- We know that

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Since  $X_s$  is white noise,

$$R_x(s) = \sigma^2 \delta(s)$$

$$S_x(e^{j\mu}, e^{j\nu}) = \sigma^2$$

- So the power spectrum of  $Y_s$  is given by

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\sigma^2}{|1 - H(e^{j\mu}, e^{j\nu})|^2}$$

## Spectral Estimate Using AR Processes

- Compute MMSE linear predictor  $\hat{h}_s$  for  $Y_s$ .
- Compute the prediction variance

$$\hat{\sigma}^2 = \frac{1}{|S|} \sum_{s \in S} |Y_s - h_s * Y_s|^2$$

where  $S$  is a finite set of points in plain, and  $|S|$  is the number of points in  $S$ .

- Estimate the power spectrum

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\hat{\sigma}^2}{|1 - \hat{H}(e^{j\mu}, e^{j\nu})|^2}$$

- Can produce a more accurate estimate of the power spectrum.



## Generating AR Processes

- Select a causal prediction filter  $h_s$ .
- Apply IIR filter to white noise random process  $X_s$

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- $Y_s$  is sometimes referred to as a white noise driven process.
- Do linear FIR prediction filters  $\hat{h}_s$  always form a stable IIR filter?
  - In 1-D, yes.
  - In 2-D, not always!
- Other problems:
  - Causal ordering of points may cause asymmetric artifacts in results.
  - Complexity increases rapidly with IIR filter order  $P$ .