

Random Variables

- Let X be a random variable on $\mathbb{R}$, then
 - X is usually denoted by an upper case letter.
 - The cumulative distribution function is given by

$$P\{X \leq x\} = F_X(x)$$

- If the probability density function exists, it is given by

$$p_X(x) = \frac{dF_X(x)}{dx}$$

so that

$$\begin{aligned} P\{x_1 < X \leq x_2\} &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} p_X(\tau) d\tau \end{aligned}$$

- The expectation of X is given by

$$E[X] = \int_{-\infty}^{\infty} \tau p_X(\tau) d\tau$$

or more precisely by the Riemann-Stieltjes integral

$$E[X] = \int_{-\infty}^{\infty} \tau dF_X(\tau)$$

if it exists.

Deterministic versus Random

- Let X and Z be random variables, and let $f(\cdot)$ be a function from $\mathcal{R}$ to $\mathcal{R}$

- Is Y a random variable

$$Y = f(X)$$

- Is μ a random variable

$$\mu = E[X]$$

- Is $\hat{X}$ a random variable

$$\hat{X} = E[X|Z]$$

Properties of Expectation

- Expectation is linear

$$E[X + Y] = E[X] + E[Y]$$

- What is $E[E[X|Y]]$ equal to?

$$E[E[X|Y]] = E[X]$$

- What is $E[X|X, Y]$ equal to?

$$E[X|X, Y] = X$$

- When X , Y , and Z are (jointly) Gaussian

$$E[X|Y, Z] = aY + bZ + c$$

for some scalar values a , b , and c .

2-D Discrete Space Random Processes

- Notation

- X_s is a pixel at position $s = (s_1, s_2) \in \mathcal{Z}^2$
- S denotes the set of 2-D Lattice points where $S \subset \mathcal{Z}^2$

- Definitions

- Mean $\mu_s = E[X_s]$
- Autocorrelation $R_{sr} = E[X_s X_r]$
- Autocovariance $C_{sr} = E[(X_s - \mu_s)(X_r - \mu_r)]$
- A process is said to be **second order** if $E[X_s]$ and $E[X_s X_r]$ exist for all $s \in S$ and $r \in S$.
- A second order random process is said to be **wide sense stationary** if for all $s \in \mathcal{Z}^2$

$$\mu_s = \mu_{(0,0)}$$

$$C_{r,r+s} = C_{(0,0),s}$$

2-D Power Spectral Density

Let X_s be a zero mean wide sense stationary random process.

Define

$$\hat{X}_N(e^{j\mu}, e^{j\nu}) = \sum_{m=-N}^N \sum_{n=-N}^N X_{(m,n)} e^{-j(m\mu+n\nu)}$$

- Then the power spectrum (i.e. energy spectrum per unit sample) is

$$\frac{1}{(2N+1)^2} |\hat{X}_N(e^{j\mu}, e^{j\nu})|^2$$

The following limit does not converge!!

$$\lim_{N \rightarrow \infty} \frac{1}{(2N+1)^2} |\hat{X}_N(e^{j\mu}, e^{j\nu})|^2$$

Intuition - The spectral estimate remains noisy as the window size increases.

Definition of Power Spectral Density

- Definition of **Power Spectral Density**

$$S_x(e^{j\mu}, e^{j\nu}) \triangleq \lim_{N \rightarrow \infty} \frac{1}{(2N+1)^2} E \left[\left| \hat{X}_N(e^{j\mu}, e^{j\nu}) \right|^2 \right]$$

Expectation removes the noise.

Weiner-Khintchine Theorem

- For a wide sense stationary random process, the power spectral density equals the Fourier transform of the autocorrelation

$$S_x(e^{j\mu}, e^{j\nu}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m, n) e^{-j(m\mu + n\nu)}$$

where

$$R(m, n) = E[X_{(0,0)} X_{(m,n)}]$$