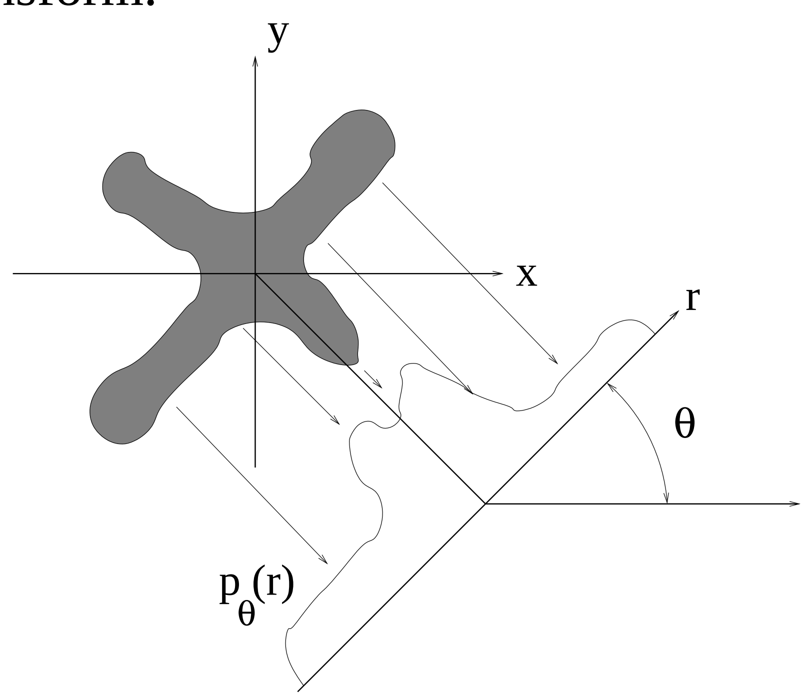
Application: Tomography

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.
  - Projections must be collected at every angle θ and displacement r.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.



- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

Coordinate Rotation

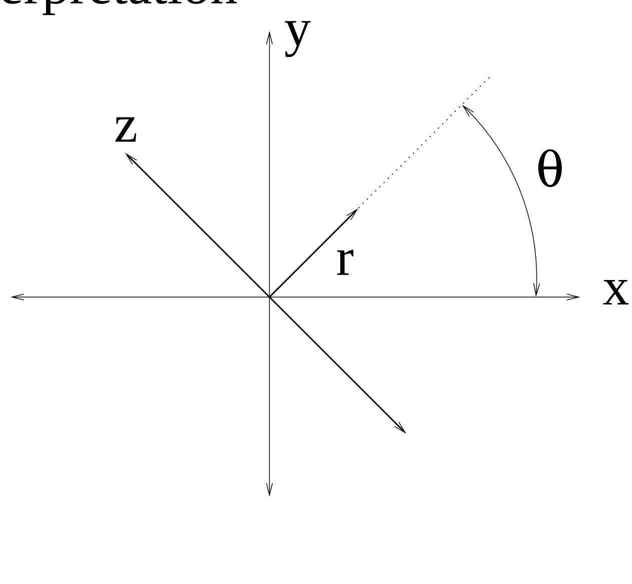
- Define the counter-clockwise rotation matrix

$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Define the new coordinate system (r, z)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$$

- Geometric interpretation

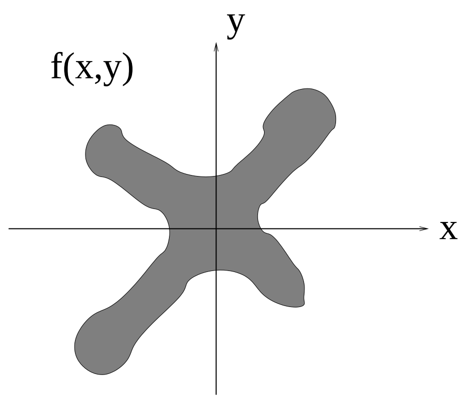


- Inverse transformation

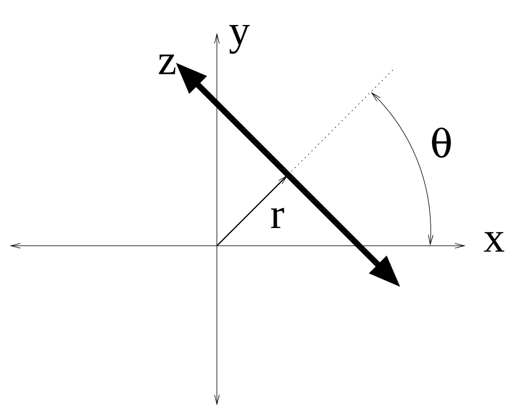
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

Integration Along Projections

- Consider the function $f(x, y)$.



- We compute projections by integrating along z for each r.



- The projection integral for each r and θ is given by

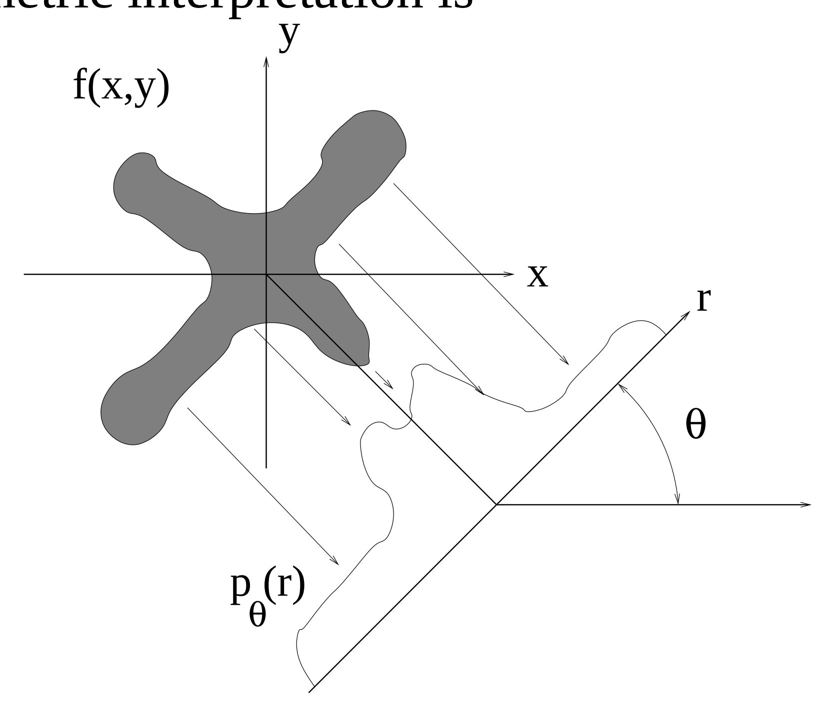
$$
\begin{aligned}
p_\theta(r) &= \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz
\end{aligned}
$$

The Radon Transform

- The Radon transform of the function $f(x,y)$ is defined as

$$p_\theta(r) = \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz$$

- The geometric interpretation is



Notice that the projection corresponding to $r = 0$ goes through the point $(x,y) = (0,0)$.

The Fourier Slice Theorem

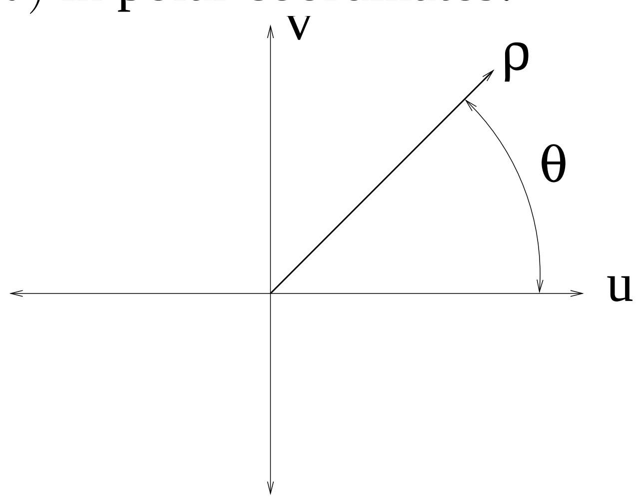
- Let

$$P_\theta(\rho) = CTFT\{p_\theta(r)\}$$
$$F(u,v) = CSFT\{f(x,y)\}$$

Then

$$P_\theta(\rho) = F(\rho\cos(\theta), \rho\sin(\theta))$$

- $P_\theta(\rho)$ is $F(u,v)$ in polar coordinates!



Proof of the Fourier Slice Theorem

- By definition

$$p_\theta(r) \; = \; \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz$$

- The CTFT of $p_\theta(r)$ is then given by

$$\begin{aligned} P_\theta(\rho) \; &= \; \int_{-\infty}^{\infty} p_\theta(r) e^{-j2\pi\rho r} dr \\ &= \; \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz\right] e^{-j2\pi\rho r} dr \\ &= \; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) e^{-j2\pi\rho r} dz dr \end{aligned}$$

- We next make the change of variables

$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} \; .$$

Notice that the Jacobian is $|\mathbf{A}_\theta| = 1$, and that $r = x\cos(\theta) + y\sin(\theta)$. This results in

$$\begin{aligned} P_\theta(\rho) \; &= \; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(x, y\right) e^{-j2\pi\rho[x\cos(\theta) + y\sin(\theta)]} dx dy \\ &= \; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(x, y\right) e^{-j2\pi[x\rho\cos(\theta) + y\rho\sin(\theta)]} dx dy \\ &= \; F(\rho\cos(\theta), \rho\sin(\theta)) \end{aligned}$$

Alternative Proof of the Fourier Slice Theorem

- First let $\theta = 0$, then

$$p_0(r) \;=\; \int_{-\infty}^{\infty} f(r,y)\, dy$$

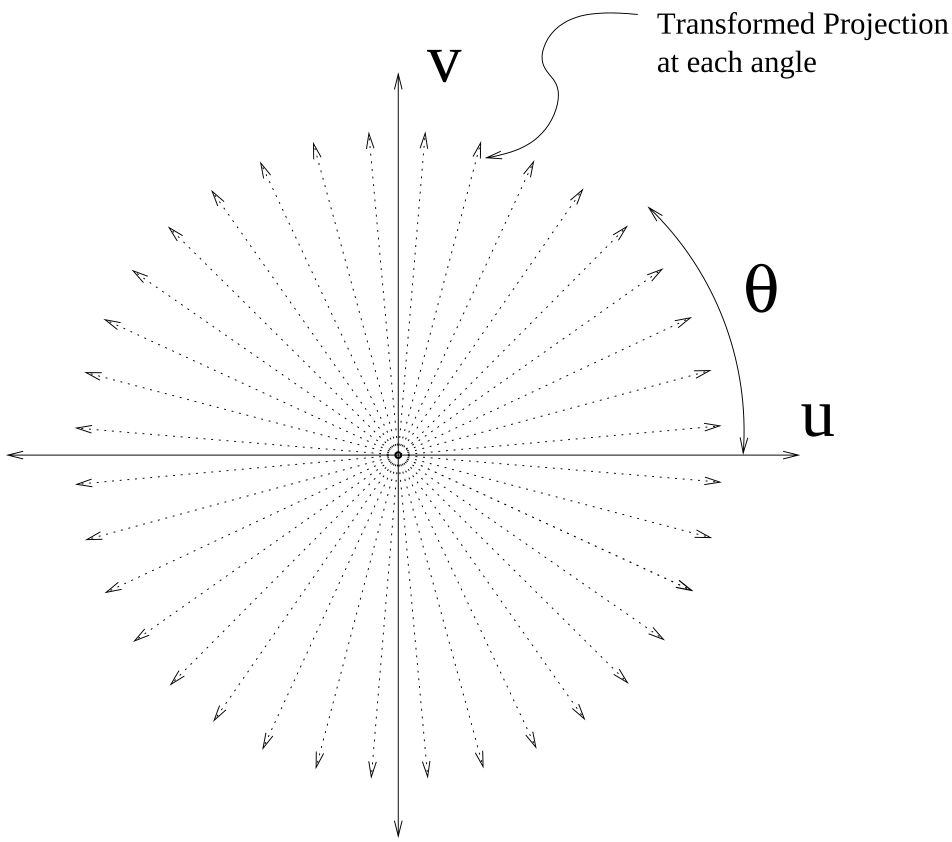
Then

$$\begin{aligned} P_0(\rho) \;&=\; \int_{-\infty}^{\infty} p_0(r) e^{-2\pi j r\rho}\, dr \\ &=\; \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(r,y)\, dy\right] e^{-2\pi j r\rho}\, dr \\ &=\; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,y) e^{-2\pi j (r\rho + y0)}\, dr\, dy \\ &=\; F(\rho, 0) \end{aligned}$$

- By rotation property of CSFT, it must hold for any θ.

Inverse Radon Transform

- Physical systems measure $p_\theta(r)$.
- From these, we compute $P_\theta(\rho) = DTFT\{p_\theta(r)\}$.



- Next we take an inverse CSFT to form $f(x, y)$.

  Problem: This requires polar to rectagular conversion.

  Solution: Convolution backprojection

Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of $F(u, v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du\, dv = |\rho| d\theta\, d\rho$$

- This results in the expression

$$\begin{aligned} f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi j(xu+yv)}\, dudv \\ &= \int_{-\infty}^{\infty} \int_{0}^{\pi} P_\theta(\rho) e^{2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))}\, |\rho| d\theta\, d\rho \\ &= \int_{0}^{\pi} \underbrace{\left[\int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho(x\cos(\theta)+y\sin(\theta))}\, d\rho \right]}_{g_\theta(x\cos(\theta)+y\sin(\theta))} d\theta \end{aligned}$$

- Then $g(t)$ is given by

$$\begin{aligned} g_\theta(t) &= \int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho t}\, d\rho \\ &= CTFT^{-1}\left\{ |\rho| P_\theta(\rho) \right\} \\ &= h(t) * p_\theta(r) \end{aligned}$$

where $h(t) = CTFT^{-1}\{|\rho|\}$, and

$$f(x, y) = \int_{0}^{\pi} g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

Summary of CBP Algorithm

1. Measure projections $p_\theta(r)$.
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.
3. Back project filtered projections

$$f(x, y) = \int_0^\pi g_\theta \left(x \cos(\theta) + y \sin(\theta)\right) d\theta$$

A Closer Look at Projection Filter

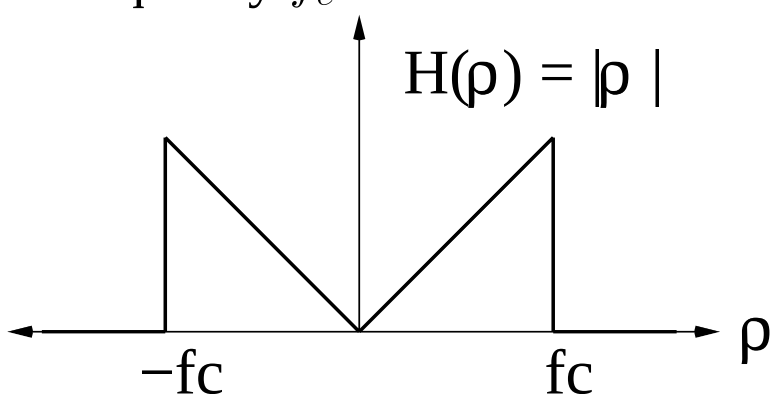
1. At each angle, projections are filtered.

$$g_\theta(r) = h(r) * p_\theta(r)$$

2. The frequency response of the filter is given by

$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to $|\rho| \leq f_c$ for some cut-off frequency f_c.



So

$$\begin{aligned} H(\rho) &= f_c \left[\text{rect}\left(f/(2f_c)\right) - \Lambda\left(f/f_c\right)\right] \\ h(r) &= f_c^2 \left[2\text{sinc}(t2f_c) - \text{sinc}^2(tf_c)\right] \end{aligned}$$

A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^\pi b_\theta(x, y)\, d\theta$$

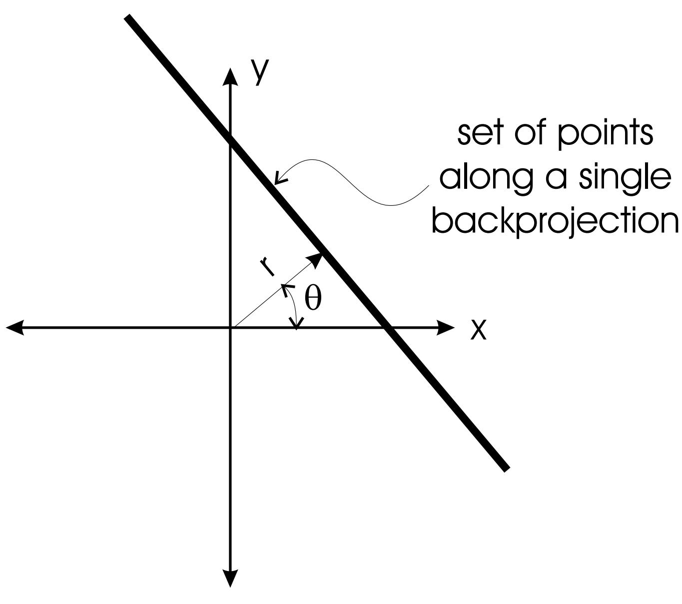
where

$$b_\theta(x, y) = g_\theta(x\cos(\theta) + y\sin(\theta))$$

- Consider the set of points (x, y) such that

$$r = x\cos(\theta) + y\sin(\theta)$$

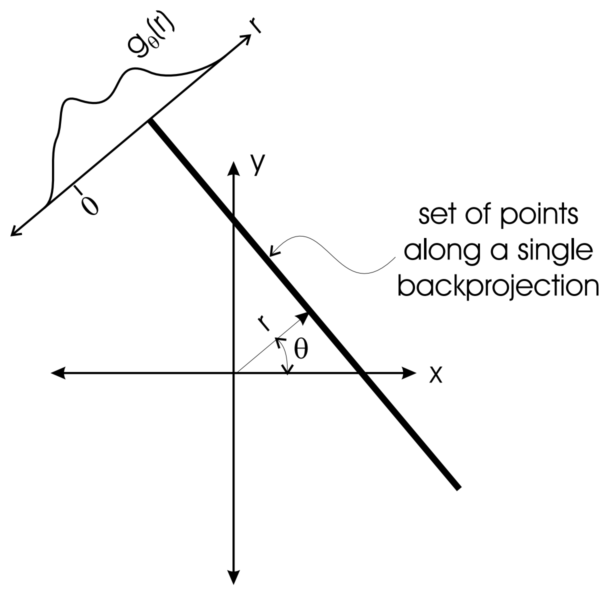
This set looks like



- Along this line $b_\theta(x, y) = g_\theta(r)$.

Back Projection Continued

- For each angle θ back projection is constant along lines of angle θ and takes on value $g_\theta(r)$.



- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to π.

$$f(x,y) = \int_0^\pi b_\theta(x,y)\,d\theta$$

$$\approx \frac{\pi}{M}\sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x,y)$$

- Back projection "smears" values of $g(r)$ back over image, and then adds smeared images for each angle.