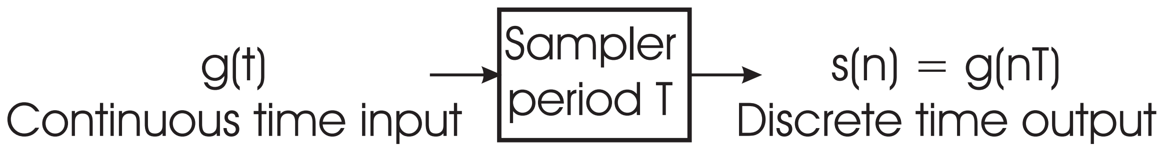
1-D Sampling



- Let $f_s = 1/T$ be the sampling frequency.
- What is the relationship between $S(e^{j\omega})$ and $G(f)$?

$$S(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

- Intuition
  - Scale frequencies

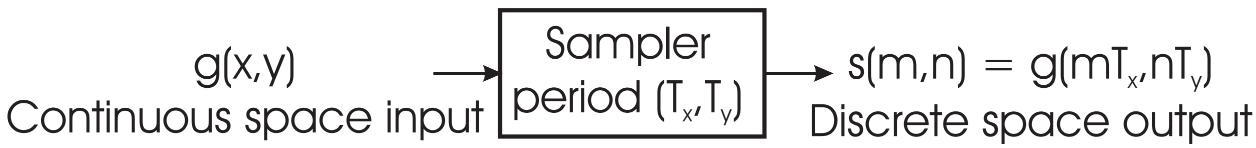
$$f = 0 \quad \Leftrightarrow \quad \omega = 0$$

$$f = \frac{1}{2T} = \frac{1}{2} f_s \quad \Leftrightarrow \quad \omega = \pi$$

$$f = \frac{1}{T} = f_s \quad \Leftrightarrow \quad \omega = 2\pi$$

  - Replicate at period 2π
  - Apply gain factor of $\frac{1}{T}$.

2-D Sampling



- Let T_x and T_y be the sampling period in the x and y dimensions.
- Then

$$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T_x T_y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G\left(\frac{\mu - 2\pi k}{2\pi T_x}, \frac{\nu - 2\pi l}{2\pi T_y} \right)$$

- Intuition
  - Scale frequencies

$$(u, v) = (0, 0) \quad \Leftrightarrow \quad (\mu, \nu) = (0, 0)$$

$$(u, v) = (\frac{1}{2T_x}, 0) \quad \Leftrightarrow \quad (\mu, \nu) = (\pi, 0)$$

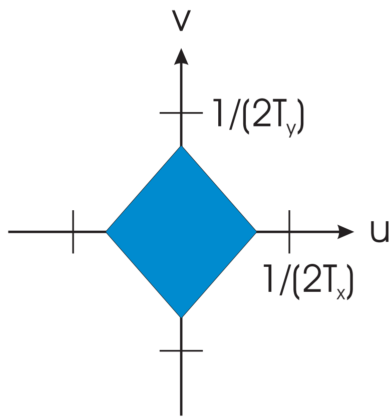
$$(u, v) = (0, \frac{1}{2T_y}) \quad \Leftrightarrow \quad (\mu, \nu) = (0, \pi)$$

$$(u, v) = (\frac{1}{2T_x}, \frac{1}{2T_y}) \quad \Leftrightarrow \quad (\mu, \nu) = (\pi, \pi)$$

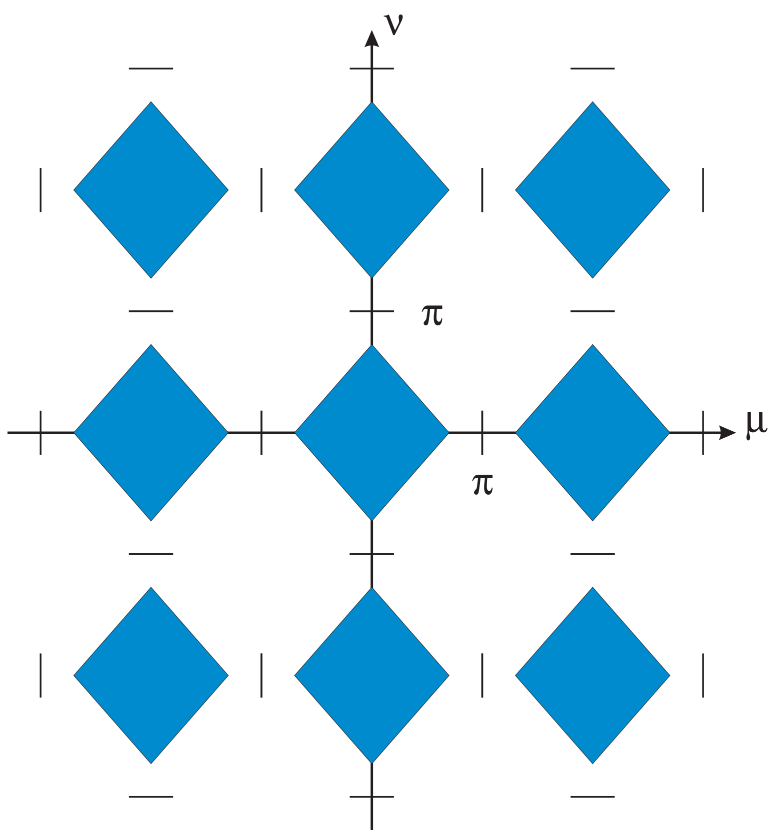
  - Replicate along both μ and ν with period 2π
  - Apply gain factor of $\frac{1}{T_x T_y}$.

Example 1: 2-D Sampling Without Aliasing

$G(u, v)$ - Spectrum of continuous space image.

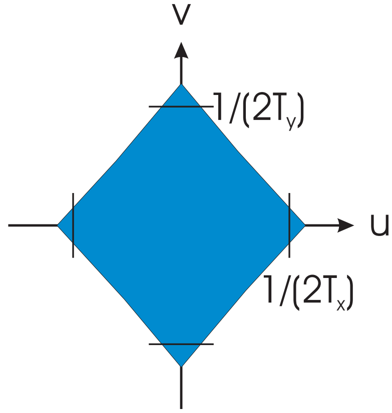


$S(e^{j\mu}, e^{j\nu})$ - Spectrum of sampled image.

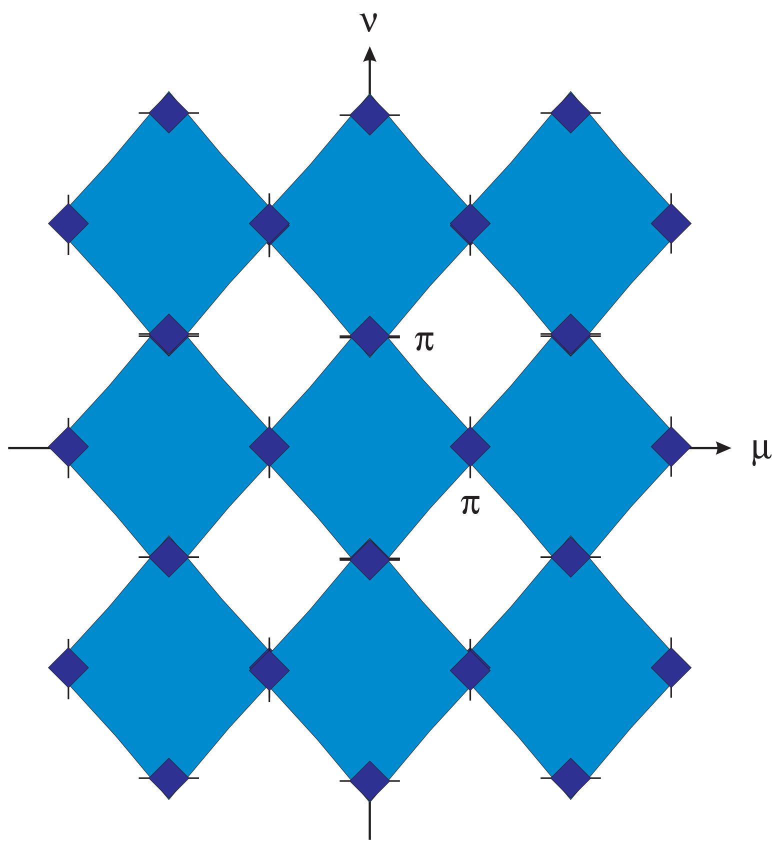


Example 2: 2-D Sampling With Aliasing

$G(u, v)$ - Spectrum of continuous space image.



$S(e^{j\mu}, e^{j\nu})$ - Spectrum of sampled image.



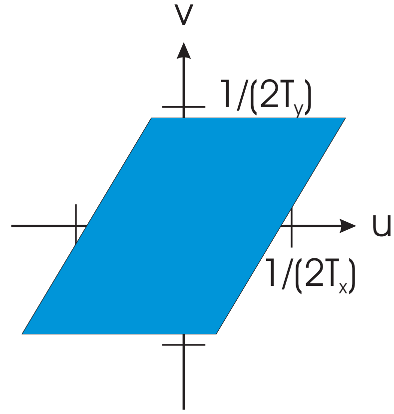
Nyquist Condition

- A continuous-space signal, $g(x, y)$, may be uniquely reconstructed from its sampled version, $s(m, n)$, if $G(u, v) = 0$ for all $|u| > \frac{1}{2T_x}$ and $|v| > \frac{1}{2T_y}$.

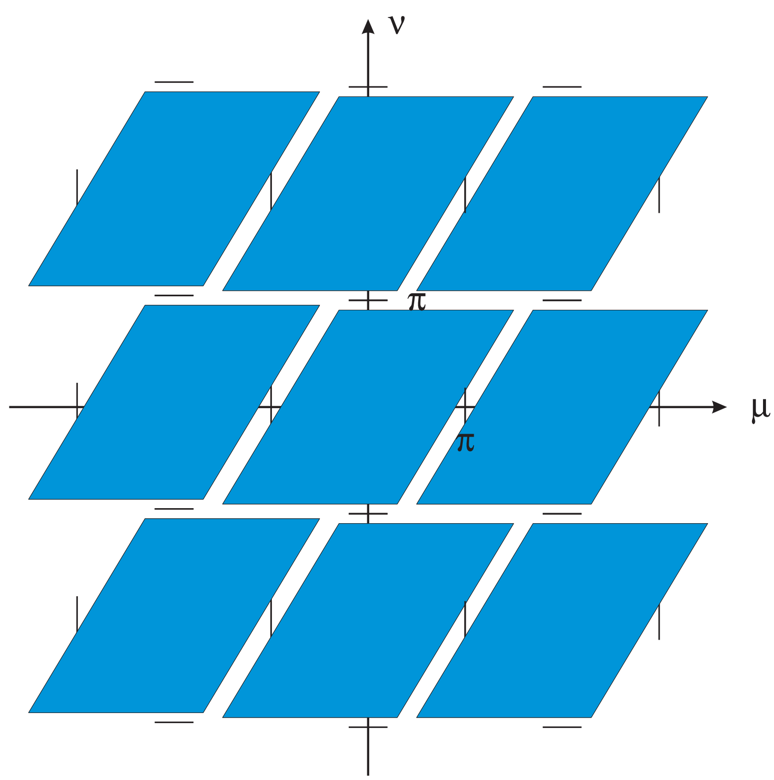
- This condition is sufficient, but not necessary.

Example 3: Nonrectangular Spectral Support

$G(u, v)$ - Spectrum of continuous space image.

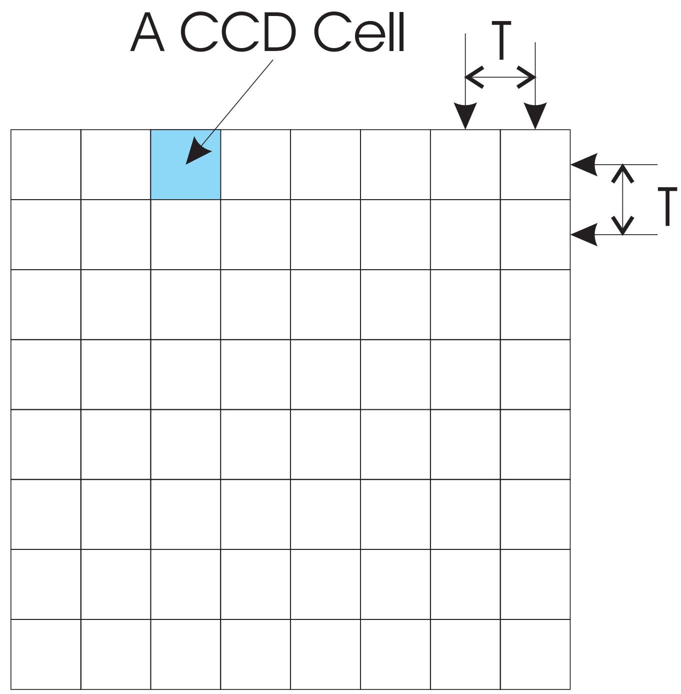


$S(e^{j\mu}, e^{j\nu})$ - Spectrum of sampled image.



Focal Plain Arrays

- Typical Charged Coupled Devices (CCD) Imaging array



- Solid state device used in video and still cameras.
- Each cell collects photons in a square $T \times T$ region.
- Response of each cell is linear with energy (photons).
- Signal is "shifted out" after captured.
- Cell should be large for best sensitivity.
- Finite cell size violates sampling assumptions.

Mathematical Model for CCD

- Let $s(m,n)$ be the output of cell (m,n), then

$$s(m,n) = \int_{\mathbb{R}^2} h(x-mT, y-nT)\, g(x,y)\, dxdy$$

where $h(x,y)$ is the rectangular window for each cell.

$$h(x,y) = \frac{1}{T^2}\mathrm{rect}(x/T, y/T)$$

- Define $\tilde{g}(x,y)$ so that

$$\begin{aligned}\tilde{g}(\xi,\eta) &= \int_{\mathbb{R}^2} h(x-\xi, y-\eta)\, g(x,y)\, dxdy \\ &= h(-x,-y) * g(x,y)\end{aligned}$$

and then we have that

$$s(m,n) = \tilde{g}(mT, nT)$$

CCD Model in Space Domain

- Filter signal with space reversed cell profile

$$\begin{aligned}\tilde{g}(x,y) &= h(-x,-y) * g(x,y) \\ &= \frac{1}{T^2}\mathrm{rect}(x/T, y/T) * g(x,y)\end{aligned}$$

- Sample filtered image

$$s(m,n) = \tilde{g}(mT, nT)$$

- Cell aperture blurs image.

CCD Model in Frequency Domain

- Filter signal with cell profile

$$\begin{aligned}\tilde{G}(u,v) &= H^*(u,v)G(u,v)\\ &= \text{sinc}(uT,vT)G(u,v)\end{aligned}$$

- Sample filtered image

$$S(e^{j\mu},e^{j\nu}) = \frac{1}{T^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\tilde{G}\left(\frac{\mu-2\pi k}{2\pi T},\frac{\nu-2\pi l}{2\pi T}\right)$$

- Complete model

$$\begin{aligned}S(e^{j\mu},e^{j\nu}) = \frac{1}{T^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\text{sinc}\left(\frac{\mu-2\pi k}{2\pi},\frac{\nu-2\pi l}{2\pi}\right)\cdot\\ G\left(\frac{\mu-2\pi k}{2\pi T},\frac{\nu-2\pi l}{2\pi T}\right)\end{aligned}$$

- Sinc function filters image.

Sampled Image Display or Rendering (Reconstruction)

- CRT's and LCD displays convert discrete-space images to continuous-space images.
- Notation:

  $s(m, n)$ - sampled image
  $p(x, y)$ - point spread function (PSF) of display
  $f(x, y)$ - displayed image

- Model:
  - In space domain:

    $$f(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(m, n) p(x - mT, y - nT)$$

  - In frequency domain:

    $$F(u, v) = P(u, v) S(e^{j2\pi Tu}, e^{j2\pi Tv})$$

    $$\mu \rightarrow 2\pi Tu$$

    $$\nu \rightarrow 2\pi Tv$$

- Monitor PSF further "softens" image.

Model for Sampling and Reconstruction

- Combining models for sampling and reconstruction results in:

$$F(u,v) =$$

$$\frac{P(u,v)}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} H^*\left(u - \frac{K}{T}, v - \frac{l}{T}\right) G\left(u - \frac{K}{T}, v - \frac{l}{T}\right)$$

- When no aliasing occurs, this reduces to

$$\begin{aligned} F(u,v) &= \frac{P(u,v)H^*(u,v)}{T^2} G(u,v) \\ &= \frac{P(u,v)\,\mathrm{sinc}(uT,vT)}{T^2} G(u,v) \end{aligned}$$

Effect of Sampling and Reconstruction

- The image is effectively filtered by the transfer function
$$\frac{1}{T^2}P(u,v)H^*(u,v) = \frac{1}{T^2}P(u,v)\mathrm{sinc}(uT,vT)$$
- Scanned image normally must be “sharpened” to remove the effect of softening produced in the scanning and display processes.

Raster Scan Ordering

- Specific scan pattern for mapping 2-D images to 1-D.
- Order pixels from top to bottom and left to right.
- Example: Consider the discrete-space image $f(m,n)$

$$\begin{bmatrix} f(0,0) & \cdots & f(M-1,0) \\ \vdots & \ddots & \vdots \\ f(0,N-1) & \cdots & f(M-1,N-1) \end{bmatrix}$$

- Raster ordering produces a 1-D signal x_n

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_{M-1} \\ x_M & x_{M+1} & \cdots & x_{2*M-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(N-1)*M} & x_{(N-1)*M+1} & \cdots & x_{N*M-1} \end{bmatrix}$$

Vector Representation of Images

- An image is **not a matrix**. (A Matrix specifies a linear function.)
- Vectorizing images
  - Often image must be converted to a vector (data).
  - Vector looks like

$$x = \begin{bmatrix} f(0,0) \\ \vdots \\ f(M-1,0) \\ \vdots \\ f(0,N-1) \\ \vdots \\ f(M-1,N-1) \end{bmatrix}$$

  - Mapping from vector to image is $f(m,n) = x_{n*M+m}$.