Image Restortation

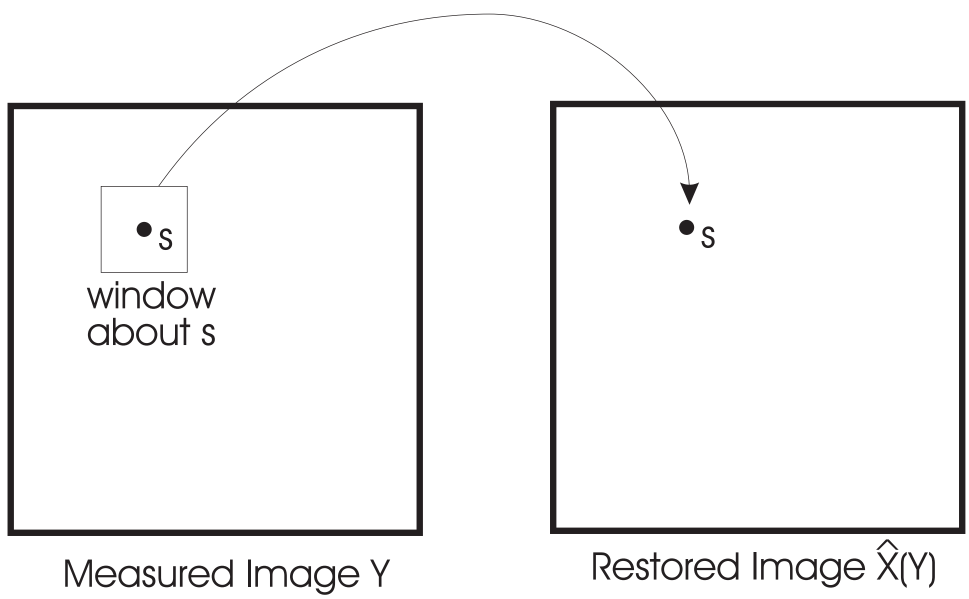
- Problem:
  - You want to know some image X.
  - But you only have a corrupted version Y.
  - How do you determine X from Y?
- Corruption may result from:
  - Additive noise
  - Nonadditive noise
  - Linear distortion
  - Nonlinear distortion

Optimum Linear FIR Filter

- Find an "optimum" linear filter to compute X from Y.
- Filter uses input window of Y to estimate each output pixel X_s.
- Filter can be designed to be minimize mean squared error (MMSE).
- The estimate of X_s is denoted by $\hat{X}_s$.
- $W(s)$ denotes the window about s.
- The estimate, $\hat{X}_s$, is a function of $Y_{W(s)}$.

Application of Optimum Filter

$$\hat{X}_s = f(Y_{W(s)})$$



- The function $f(Y_{W(s)})$ is designed to produce a MMSE estimate of X.
- If $f(Y_{W(s)})$ is:
  - Linear $\Rightarrow$ linear space invariant filter.
  - Nonlinear $\Rightarrow$ nonlinear space invariant filter.
- This filter can reduce the effects of all types of corruption.

Optimality Properities of Linear Filter

- If both images are jointly Gaussian:
  - Then MMSE filter is linear.

$$\begin{aligned}\hat{X}_s &= E[X_s|Y_{W(s)}] \\ &= \mathbf{A}\,Y_{W(s)} + b\end{aligned}$$

- If images are not jointly Gaussian:
  - Then MMSE filter is generally not linear.

$$\begin{aligned}\hat{X}_s &= E[X_s|Y_{W(s)}] \\ &= f(Y_{W(s)})\end{aligned}$$

  - However, the MMSE linear filter can still be very effective!

Formulation of MMSE Linear Filter: Definitions

- $W(s)$ - window about the pixel s.
- p - number of pixels in $W(s)$
- z_s - row vector containing pixels of $Y_{W(s)}$.
- θ - parameter vector
- Detailed definitions:
  - Definition of $W(s)$

    $$W(s) = [s, s + r_1, \ldots, s + r_{p-1}]$$

    where $r_1, \ldots, r_{p-1}$ index neighbors.
  - Definition of z_s

    $$z_s = [y_s, y_{s+r_1}, \ldots, y_{s+r_{p-1}}]$$

  - Definition of θ

    $$\theta = [\theta_0, \ldots, \theta_{p-1}]$$

Formulation of MMSE Linear Filter: Objectives

- Linear filter is given by

$$\hat{x}_s = z_s \theta$$

- Mean squared error is given by

$$\begin{aligned} MSE &= E[|x_s - \hat{x}_s|^2] \\ &= E[|x_s - z_s \theta|^2] \end{aligned}$$

- The MMSE filter parameters θ^* are given by

$$\theta^* = \arg\min_{\theta} E[|x_s - z_s \theta|^2] \ .$$

- How do we solve this problem?

More Matrix Notation

- Define the subset S_0 of image pixels.
  1. $S_0 \subset S$
  2. S_0 contains $N_0 < N$ pixels
  3. S_0 usually does not contain pixels on the boundary of the image.
  4. $S_0 = [s_1, \ldots, s_{N_0}]$
- Define the $N_0 \times p$ matrix Z

$$Z = \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} .$$

- Define the $N_0 \times 1$ column vectors X and $\hat{X}$

$$X = \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{bmatrix} \hat{x}_{s_1} \\ \hat{x}_{s_2} \\ \vdots \\ \hat{x}_{s_{N_0}} \end{bmatrix} .$$

- Then

$$X \approx \hat{X} = Z\theta$$

Least Squares Linear Filter

- We expect that

$$\begin{aligned} MSE &= E[|x_s - z_s\theta|^2] \\ &\approx \frac{1}{N_0}\sum_{s\in S_0}|x_s - z_s\theta|^2 \\ &= \frac{1}{N_0}||X - Z\theta||^2 \end{aligned}$$

- So we may solve the equation

$$\theta^* = \arg\min_{\theta} ||X - Z\theta||^2$$

- The solution θ^* is the least squares estimate, of θ, and the estimate

$$\hat{X} = Z\theta^*$$

is known as the least squares filter.

Computing Least Squares Linear Filter

$$\theta^* = \arg\min_\theta \frac{1}{N_0} ||X - Z\theta||^2$$

- So

$$\begin{aligned}
\theta^* &= \arg\min_\theta \left(\frac{1}{N_0} ||X - Z\theta||^2 \right) \\
&= \arg\min_\theta \left(\frac{1}{N_0} (X - Z\theta)^t (X - Z\theta) \right) \\
&= \arg\min_\theta \left(\frac{1}{N_0} (X^t X - 2\theta^t Z^t X + \theta^t Z^t Z\theta) \right) \\
&= \arg\min_\theta \left(\frac{X^t X}{N_0} - 2\theta^t \frac{Z^t X}{N_0} + \theta^t \frac{Z^t Z}{N_0} \theta \right) \\
&= \arg\min_\theta \left(\theta^t \frac{Z^t Z}{N_0} \theta - 2\theta^t \frac{Z^t X}{N_0} \right)
\end{aligned}$$

Covariance Estimates

- Define the $p \times p$ matrix

$$
\begin{aligned}
\hat{R}_{zz} &\triangleq \frac{Z^t Z}{N_0} \\
&= \frac{1}{N_0}\left[z_{s_1}^t, z_{s_2}^t, \ldots, z_{s_{N_0}}^t\right]\begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} \\
&= \frac{1}{N_0}\sum_{i=1}^{N_0} z_{s_i}^t z_{s_i}
\end{aligned}
$$

- Define the $p \times 1$ vector

$$
\begin{aligned}
\hat{r}_{zx} &\triangleq \frac{Z^t X}{N_0} \\
&= \frac{1}{N_0}\left[z_{s_1}^t, z_{s_2}^t, \ldots, z_{s_{N_0}}^t\right]\begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \\
&= \frac{1}{N_0}\sum_{i=1}^{N_0} z_{s_i}^t x_{s_i}
\end{aligned}
$$

- So

$$
\theta^* = \arg\min_{\theta}\left(\theta^t \hat{R}_{zz}\theta - 2\theta^t \hat{r}_{zx}\right)
$$

Interpretation of $\hat{R}_{zz}$ and $\hat{r}_{zx}$

- $\hat{R}_{zz}$ is an estimate of the covariance of z_s.

$$
\begin{aligned}
R_{zz} &\stackrel{\triangle}{=} E[z_s^t z_s] \\
&= E\left[\frac{1}{N_0}\sum_{s=1}^{N} z_s^t z_s\right] \\
&= E\left[\hat{R}_{zz}\right]
\end{aligned}
$$

- $\hat{r}_{zx}$ is an estimate of the cross correlation between z_s and x_s.

$$
\begin{aligned}
r_{zx} &\stackrel{\triangle}{=} E[z_s^t x_s] \\
&= E\left[\frac{1}{N_0}\sum_{s=1}^{N} z_s^t x_s\right] \\
&= E\left[\hat{r}_{zx}\right]
\end{aligned}
$$

Solution to Least Squares Linear Filter

- We need

$$\theta^* = \arg\min_\theta \frac{1}{N_0} ||X - Z\theta||^2$$

We have shown this is equivalent to

$$\theta^* = \arg\min_\theta \left(\theta^t \hat{R}_{zz}\theta - 2\theta^t \hat{r}_{zx}\right)$$

- Taking the gradient of the cost functional

$$\begin{aligned} 0 &= \nabla_\theta \left(\theta^t \hat{R}_{zz}\theta - 2\theta^t \hat{r}_{zx}\right)\Big|_{\theta=\theta^*} \\ &= \left(2\hat{R}_{zz}\theta - 2\hat{r}_{zx}\right)\Big|_{\theta=\theta^*} \end{aligned}$$

Solving for θ^* yeilds

$$\theta^* = \left(\hat{R}_{zz}\right)^{-1} \hat{r}_{zx}$$

Summary of Solution to Least Squares Linear Filter

- First compute

$$\hat{R}_{zz} = \frac{1}{N_0} \sum_{s=1}^{N} z_s^t z_s$$

$$\hat{r}_{zx} = \frac{1}{N_0} \sum_{s=1}^{N} z_s^t x_s$$

- Then compute

$$\theta^* = \left(\hat{R}_{zz}\right)^{-1} \hat{r}_{zx}$$

- The vector θ^* then contains the values of the filter coefficients.

Training

- θ^* is usually estimated from "training" data.
- Training data
  - Generally consists of image pairs (X, Y) where Y is the measured data and X is the undistorted image.
  - Should be typical of what you might expect.
  - Can often be difficult to obtain.
- Testing data
  - Also consists of image pairs (X, Y).
  - Is used to evaluate the effectiveness of the filters.
  - Should never be taken from the training data set.
- Training versus Testing
  - Performance on training data is always better than performance on testing data.
  - As the amount of training data increases, the performance on training and testing data both approach the best achievable performance.

Comments

- Wiener filter is the MMSE **linear** filter.
- Wiener filter may be optimal, but it isn't always good.
  - Linear filters blur edges
  - Linear filters work poorly with non-Gaussian noise.
- Nonlinear filters can be designed using the same methodologies.

Is MMSE a Good Quality Criteria for Images?

- In general, NO! ... But sometimes it is OK.
- For achromatic images, it is best to choose X and Y in L^* or gamma corrected coordinates.
- Let H be a filter that implements the CSF for the human visual system.
  - Then a better metric of error is

$$\begin{aligned} HVSE &= \|H(X-\hat{X})\|^2 \\ &= \left(X-\hat{X}\right)^t H^t H \left(X-\hat{X}\right) \\ &= \|X-\hat{X}\|_B^2 \end{aligned}$$

    where $B = H^t H$.
  - $\|X-\hat{X}\|_B^2$ is a quadratic norm.
- What is the minimum HVSE estimate $\hat{X}$?

Answer

- The answer is $\hat{X} = E[X|Y]$.
  - This is the same as for mean squared error!
  - The conditional expectation minimizes any quadratic norm of the error.
  - This is also true for non-Gaussian images.
- Let $\hat{X} = AY_{W(s)} + b$ be the MMSE **linear** filter.
  - This filter is also the minimum HVSE **linear** filter.
  - This is also true for non-Gaussian images.

Proof

- Define $V \stackrel{\triangle}{=} HX$

$$
\begin{aligned}
& \min_{\hat{X}} E\left[||X - \hat{X}||_B^2\right] \\
= & \min_{\hat{X}} E\left[||H(X - \hat{X})||^2\right] \\
= & \min_{\hat{V}} E\left[||V - \hat{V}||^2\right] \\
= & E\left[||V - E[V|Y]||^2\right] \\
= & E\left[||HX - E[HX|Y]||^2\right] \\
= & E\left[||H(X - E[X|Y])||^2\right] \\
= & E\left[||X - E[X|Y]||_B^2\right]
\end{aligned}
$$

- So, $\hat{X} = E[X|Y]$ minimizes the error measure.

$$HVSE = ||X - \hat{X}||_B^2 \ .$$