Continuous Space Fourier Transform (CSFT)

Forward CSFT:

$$F(u,v) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

Inverse CSFT:

$$f(x,y) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- Space coordinates:
  1. Usually, x is horizontal and y is vertical coordinate
  2. Usually, y points down
  3. Raster order - Television *scans* rapidly from left to right and more slowly from top to bottom.
- Frequency coordinates:
  1. u corresponds to horizontal frequency components (vertical strips).
  2. v corresponds to vertical frequency components (horizontal strips).

Useful Continuous Space Signal Definitions

$$\delta(x,y) \triangleq \delta(x)\,\delta(y)$$

$$\text{rect}(x,y) \triangleq \text{rect}(x)\,\text{rect}(y)$$

$$\text{sinc}(x,y) \triangleq \text{sinc}(x)\,\text{sinc}(y)$$

$$\text{circ}(x,y) \triangleq \text{rect}(\sqrt{x^2+y^2})$$

- A 2-D function $f(x,y)$ is said to be **separable** if it is formed by the product of two 1-D functions.

  $$f(x,y) = g(x)\,h(y)$$

  $\text{rect}(x,y)$, $\text{sinc}(x,y)$, and $\delta(x,y)$ are separable functions.

- Is $\text{circ}(x,y)$ a separable function?

CSFT Properties Inherited from CTFT

- Some properties of the CSFT are very similar to corresponding CTFT properties.

Property	Space Domain Function	CSFT
Linearity	$af(x,y) + bg(x,y)$	$aF(u,v) + bG(u,v)$
Conjugation	$f^*(x,y)$	$F^*(-u,-v)$
Scaling	$f(ax,by)$	$\frac{1}{\lvert ab\rvert}F(u/a,v/b)$
Shifting	$f(x-x_0,y-y_0)$	$e^{-j2\pi(ux_0+vy_0)}F(u,v)$
Modulation	$e^{j2\pi(u_0x+v_0y)}f(x,y)$	$F(u-u_0,v-v_0)$
Convolution	f(x,y) * g(x,y)	F(u,v) G(u,v)
Multiplication	f(x,y) g(x,y)	F(u,v) * G(u,v)

- Inner product property

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g^*(x,y)dx\,dy \\ =\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)G^*(u,v)du\,dv \end{aligned}$$

Properties Specific to CSFT

- But some properties of the CSFT are quite unique to the 2-dimensional problem.

Property	Space Domain Function	CSFT
Separability	$f(x)g(y)$	$F(u)G(v)$
Rotation	$f\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$	$\lvert\mathbf{A}\rvert^{-1}F\left([u,v]\mathbf{A}^{-1}\right)$

Separability of CSFT

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx\right]e^{-j2\pi vy}dy$$

Define the CTFT of $f(x,y)$ in the variable x

$$\tilde{F}(u,y) = \int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx$$

Then the CSFT may be computed as the CTFT of $\tilde{F}(u,y)$ in y

$$F(u,v) = \int_{-\infty}^{\infty} \tilde{F}(u,y)e^{-j2\pi vy}dy$$

- Comment: 2-D CSFT can be computed as two 1-D CTFT's.

CSFT of Separable Functions

Let

$$g(t) \overset{CTFT}{\Leftrightarrow} G(f)$$
$$h(t) \overset{CTFT}{\Leftrightarrow} H(f)$$

Then

$$g(x)h(y) \overset{CSFT}{\Leftrightarrow} G(u)H(v)$$

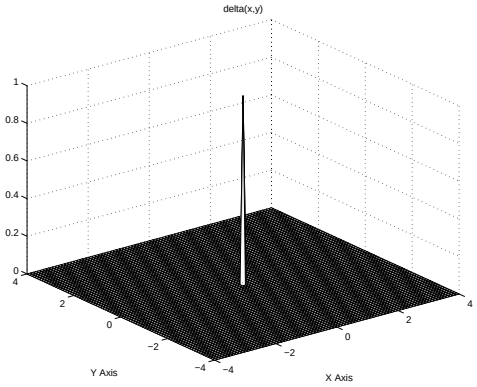
Proof:

$$\begin{aligned} F(u,v) &= CSFT\{g(x)\,h(y)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,h(y)\,e^{-j2\pi(ux+vy)}dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,h(y)\,e^{-j2\pi ux}e^{-j2\pi vy}dxdy \\ &= \left[\int_{-\infty}^{\infty} g(x)e^{-j2\pi ux}dx\right]\left[\int_{-\infty}^{\infty} h(y)e^{-j2\pi vy}dy\right] \\ &= G(u)H(v) \end{aligned}$$

Useful CSFT Transform Pairs

- 2-D delta function:

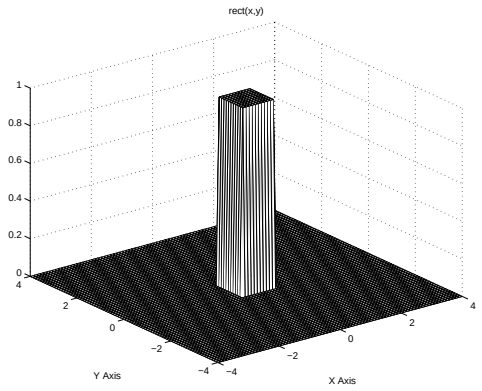
$$\begin{aligned} CSFT\left\{\delta(x,y)\right\} &= CSFT\left\{\delta(x)\delta(y)\right\} \\ &= CTFT\left\{\delta(x)\right\} \cdot CTFT\left\{\delta(y)\right\} \\ &= 1 \cdot 1 = 1 \end{aligned}$$



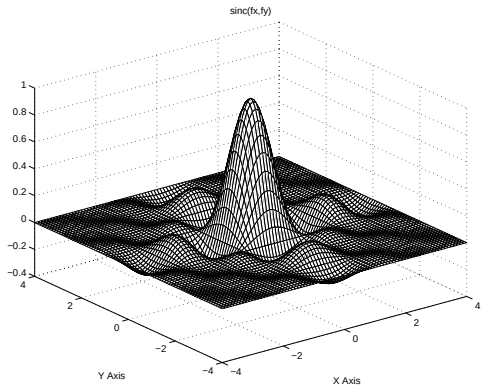
$\Rightarrow \qquad 1$

- 2-D rect function:

$$\begin{aligned} CSFT\left\{\mathrm{rect}(x,y)\right\} &= CSFT\left\{\mathrm{rect}(x)\mathrm{rect}(y)\right\} \\ &= CTFT\left\{\mathrm{rect}(x)\right\} \cdot CTFT\left\{\mathrm{rect}(y)\right\} \\ &= \mathrm{sinc}(u)\,\mathrm{sinc}(v) \\ &= \mathrm{sinc}(u,v) \end{aligned}$$



$\Rightarrow$



Rotated Functions

- Let the matrix $\mathbf{A}$ be an orthonormal rotation of angle θ

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Because $\mathbf{A}$ is an orthonormal transform

$$|\mathbf{A}| = 1$$
$$\mathbf{A}^{-1} = \mathbf{A}^t$$

- Then the CSFT of the function $g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is given by

$$\begin{aligned} CSFT\left\{g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right\} &= |\mathbf{A}|^{-1} G\left([u,v]\,\mathbf{A}^{-1}\right) \\ &= |\mathbf{A}|^{-1} G\left([u,v]\,\mathbf{A}^{t}\right) \\ &= G\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right) \end{aligned}$$

- So we have

$$g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right) \overset{CSFT}{\Leftrightarrow} G\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

Rotated Rect Function

- Rotated 2-D rect function:

$$\text{rect}\left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}\right) \;=\; \text{rect}\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

  where

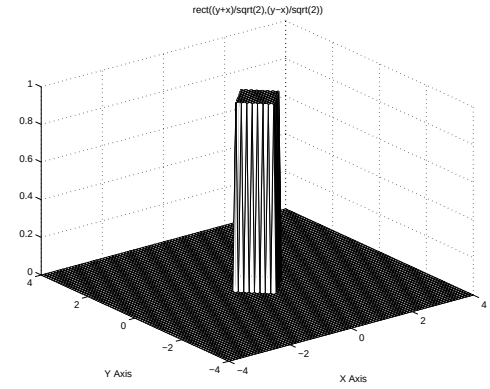
$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- $\mathbf{A}$ is a $45°$ rotation, so it is and orthonormal transform

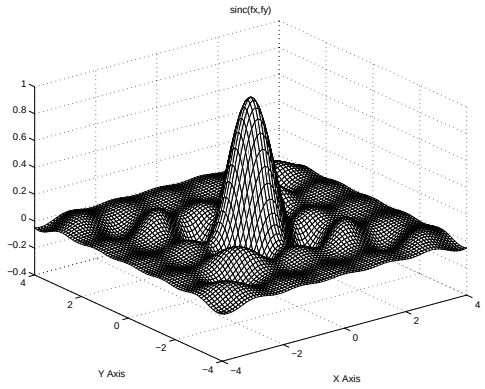
$$\begin{aligned} CSFT\left\{\text{rect}\left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}\right)\right\} &= CSFT\left\{\text{rect}\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right\} \\ &= \text{sinc}\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right) \\ &= \text{sinc}\left(\frac{v+u}{\sqrt{2}}, \frac{v-u}{\sqrt{2}}\right) \end{aligned}$$

Rotated 2-D Rect and Sinc Transform Pairs

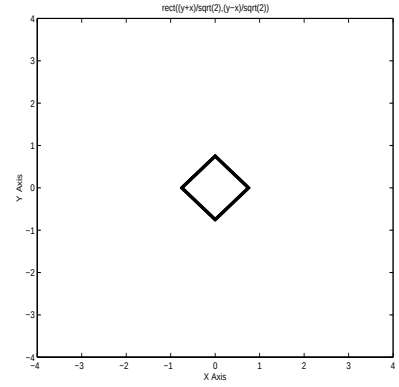
- Mesh plot



$\Rightarrow$



- Contour plot



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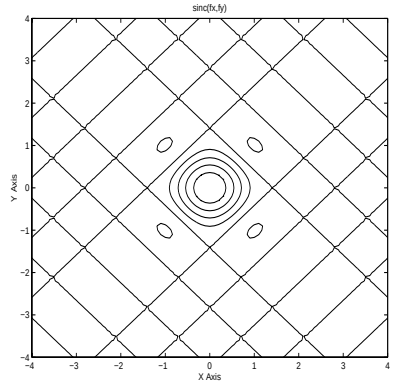


More Useful CSFT Transform Pairs

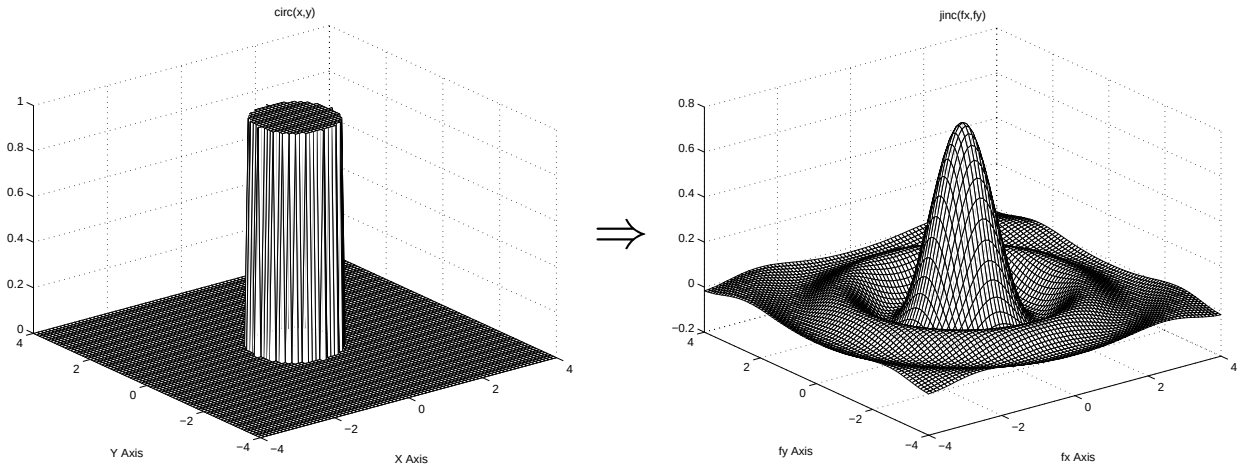
- Circ function:

$$CSFT\left\{\text{circ}(x,y)\right\} = \text{jinc}(u,v)$$

where

$$\text{jinc}(u,v) = \frac{J_1\left(\pi\sqrt{u^2+v^2}\right)}{2\sqrt{u^2+v^2}}$$

and $J_1(r)$ is the Bessel function of the first kind order 1.



- Notice that both functions are circularly symmetric

CSFT of a Plane Wave

- Consider an impulse in the 2-D frequency domain.

$$F(u,v) = \delta(u-u_o, v-v_o)$$

- Its inverse transform is a 2-D plane wave.

$$\begin{aligned} f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u-u_o, v-v_o) e^{j2\pi(ux+vy)} du dv \\ &= e^{j2\pi(u_o x + v_o y)} \end{aligned}$$

- We know that

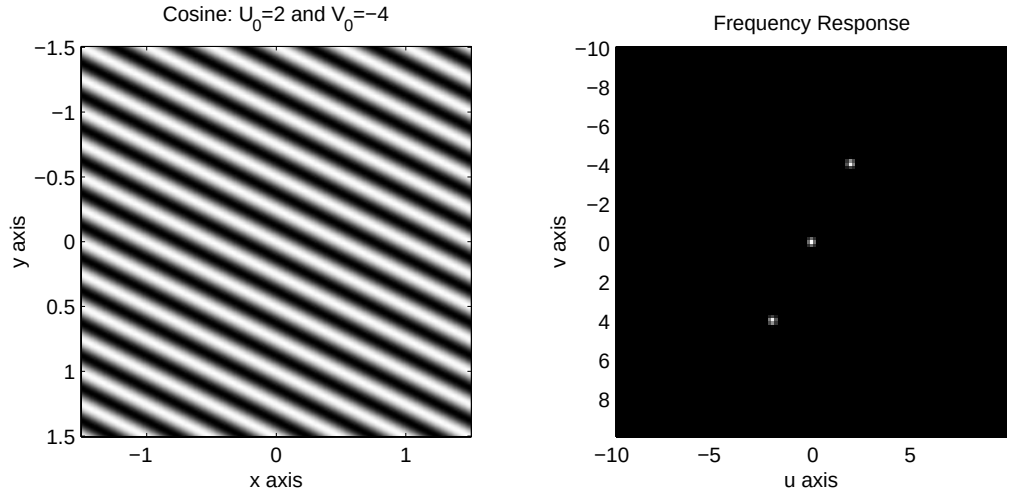
$$\cos(2\pi(u_o x + v_o y)) = \frac{1}{2}\left[e^{j2\pi(u_o x+v_o y)} + e^{-j2\pi(u_o x+v_o y)}\right]$$

- So we have that

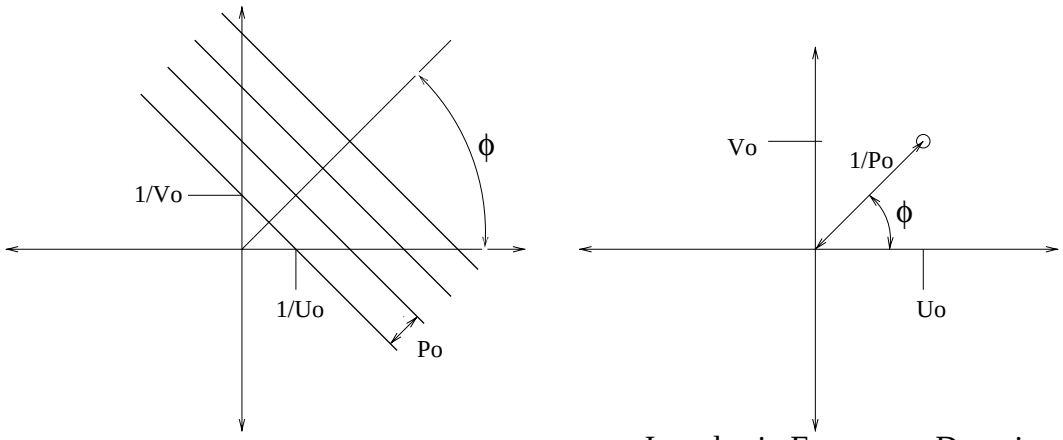
$$\begin{array}{c} \cos(2\pi(u_o x + v_o y)) \\ \overset{CSFT}{\Leftrightarrow} \\ \frac{1}{2}\left[\delta(u-u_o, v-v_o) + \delta(u+u_o, v+v_o)\right] \end{array}$$

2-D Plane Wave Example

- Example transform pair computed with Matlab [1]



- Graphical representation of space-frequency domain



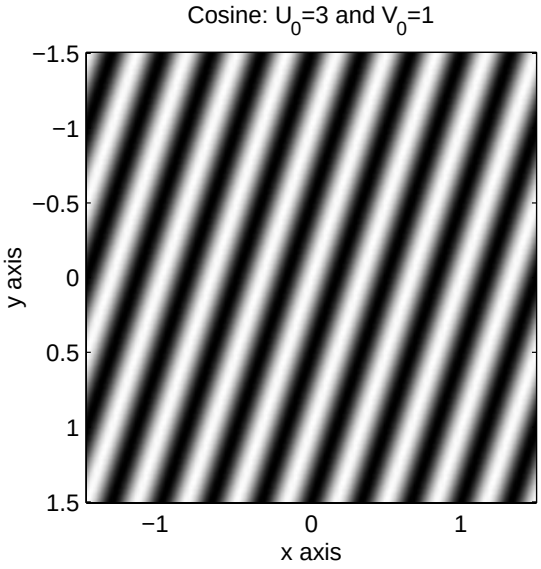
Plane Wave in Space Domain

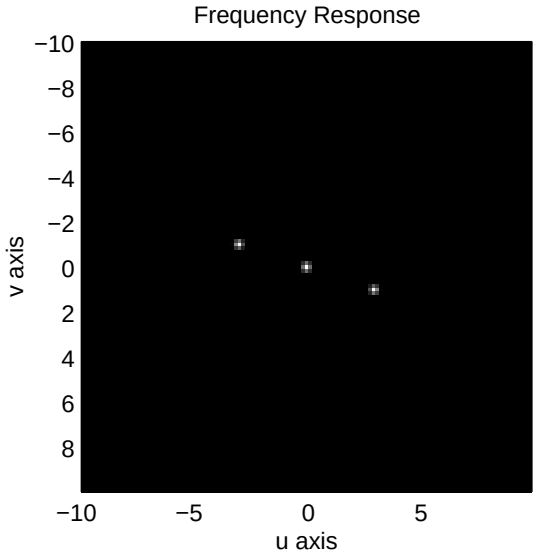
Impulse in Frequency Domain

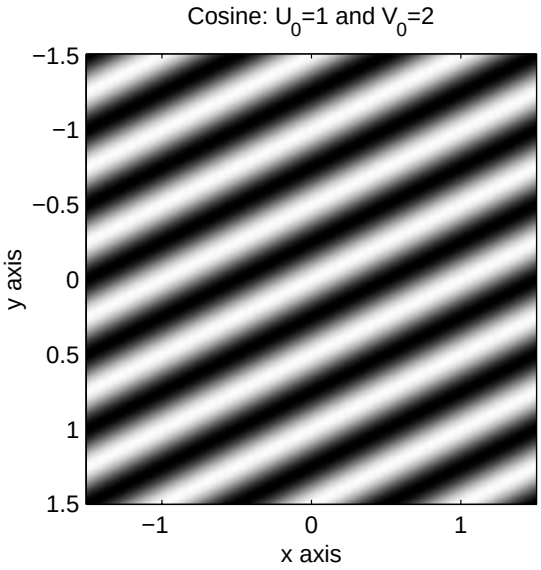
  - $1/P_0 = \sqrt{V_0^2 + U_0^2}$
  - Rotations in space and frequency domains are the same.

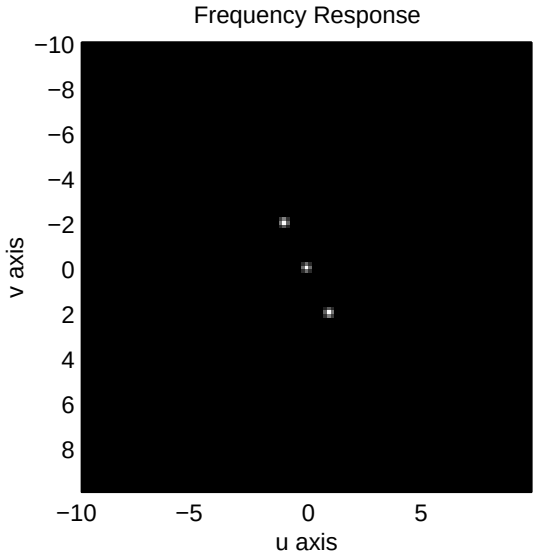
[1] $f(x, y) = \cos(u_0\, x + v_0\, y) + 0.5$

More Examples 2-D Plane Waves









More Examples 2-D Plane Waves