EE 637 Midterm Exam #1

February 20, Spring 2002

Name: ______________________________

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **three** problems.
- You may **only** use your brain and a pencil (or pen) to complete this exam. You **may not** use your book, notes or a calculator.
- This is also homework #1. Please keep it. Complete it as a homework, and hand it in by Friday at 5:00PM.

Good Luck.

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Problem 1.(25pt)

Consider the DT random process $X(n)$ formed by i. i. d. random variables with

$$P\{X(n) = 1\} = P\{X(n) = -1\} = 0.5$$

Also consider the DT random process

$$Y(n) = \sum_{k=-\infty}^{\infty} X(k)p(n-k)$$

where $p(n)$ is a real valued DT function with a DTFT of $P(e^{j\omega})$.

a) Is $Y(n)$ a stationary random process? Justify your answer.

b) Compute $R_x(k)$ autocorrelation and $S_x(e^{j\omega})$ the powerspectrum for $X(n)$.

c) Compute $R_y(k)$ autocorrelation and $S_y(e^{j\omega})$ the powerspectrum for $Y(n)$.

a) Yes

- $X(n)$ is stationary because it is an i.i.d. process
- $Y(n) = \underbrace{X(n) * p(n)}_{\text{LTI system}}$,

$\Rightarrow y(n)$ is stationary

b) $R_X(k) = \delta(k) \Rightarrow S_X(e^{j\omega}) = 1$

c) $R_Y(k) = \delta(k) * p(k) * p(-k)$

$R_Y(k) = p(k) * p(-k)$

$R_Y(k) = \sum_{\ell=-\infty}^{\infty} p(\ell)\, p(\ell + k)$

(over)

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$$S_y(e^{jw}) = P(e^{jw})\,P^*(e^{jw})$$
$$= |P(e^{jw})|^2$$

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Problem 2.(50pt)

The forward projection is defined by the operation

$$\begin{aligned} p_\theta(r) &= \mathcal{FP}\{f(x,y)\} \\ &= \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta))\, dz \end{aligned}$$

The back projection is defined by the operation

$$\begin{aligned} h(x,y) &= \mathcal{BP}\{g_\theta(r)\} \\ &= \int_0^{\pi} g_\theta(x\cos(\theta) + y\sin(\theta))d\theta \end{aligned}$$

Furthermore, define the composition of forward and back projection as the 2-D continuous space system $\mathcal{T}\{\cdot\}$

$$\begin{aligned} h(x,y) &= \mathcal{T}\{f(x,y)\} \\ &= \mathcal{BP}\{\mathcal{FP}\{f(x,y)\}\} \end{aligned}$$

a) Compute

$$p_\theta(r) = \mathcal{FP}\{\delta(x,y)\}$$

the forward projection of the signal $f(x,y) = \delta(x,y)$.

b) Show that

$$\frac{1}{\sqrt{x^2+y^2}} = \mathcal{T}\{\delta(x,y)\} \ .$$

(Hint: Use the transformation of variables $x = r\cos\phi$ and $y = r\sin\phi$.)

c) Determine

$$w(x,y) = \mathcal{T}\{\delta(x - x_0, y - y_0)\} \ .$$

(Hint: You may want to make a graphical argument.)

d) Is the system $\mathcal{T}\{\cdot\}$ linear? Justify your answer.

e) Is the system $\mathcal{T}\{\cdot\}$ space invariant? Justify your answer.

f) Propose a method for inverting the system $\mathcal{T}\{\cdot\}$.

a) $p_\theta(r) = \mathcal{FP}\{\delta(x,y)\}$

$= \delta(r)$

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b) $\mathcal{T}\{\delta(x,y)\} = \mathcal{BP}\{\mathcal{FP}\{\delta(x,y)\}\}$

$= \mathcal{BP}\{\delta(r)\}$

$$= \int_0^{\pi} \delta(x\cos\theta + y\sin\theta)\,d\theta$$

define $r = \sqrt{x^2+y^2}$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

then

$x\cos\theta + y\sin\theta = r\cos(\theta-\varphi)$

$$= \int_0^{\pi} \delta\left(r\cos(\theta-\varphi)\right)d\theta$$

$z = r\cos(\theta-\varphi)$

$\theta = \cos^{-1}\left(\frac{z}{r}\right) + \varphi$

The Jacobian of the transformation is then

$$\left|\frac{d\theta}{dz}\right| = \left|\frac{d\cos^{-1}z}{dz}\right|$$

$$= \frac{1}{r\,|z^2-1|}$$

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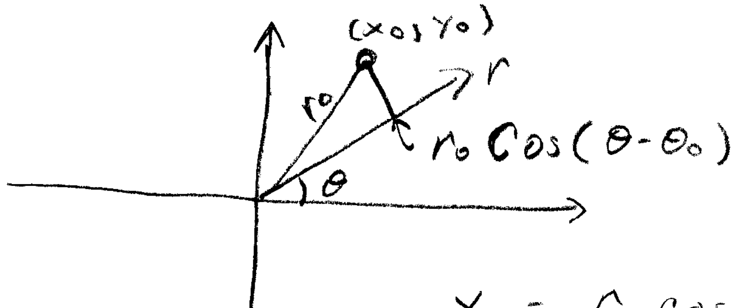
$$= \int_{-|\cos\varphi|}^{|\cos\varphi|} \delta(z) \frac{1}{r|z^2-1|} dz + 2\int_{|\cos\varphi|}^{1} \delta(z) \frac{1}{r|z^2-1|} dz$$

$$= \frac{1}{r} + 0$$

$$\mathcal{T}\{\delta(x,y)\} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$$

c)



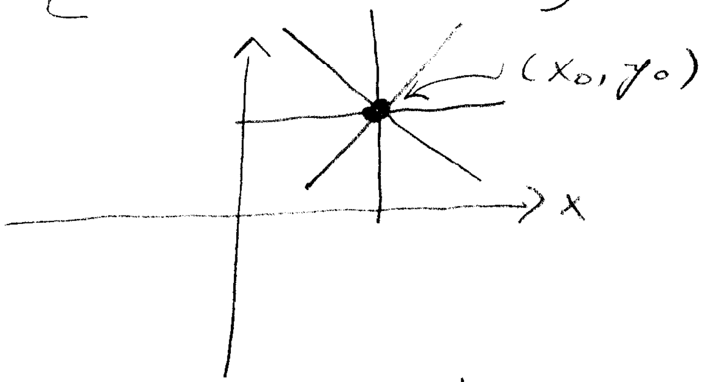
$x_0 = r_0 \cos(\theta_0)$

$y_0 = r_0 \sin(\theta_0)$

$$p_\theta(r) = \mathcal{RP}\{\delta(x_0, y_0)\}$$

$$= \delta(r - r_0 \cos(\theta))$$

$$BP\{\delta(r - r_0\cos\theta)\} = \rho_\theta(r)$$



$$\rho_\theta(r) = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

d) Yes, because the transformation is a composition of integrals, so it is linear.

e) Yes, because it is linear and its impulse response is space invariant

$$\mathcal{T}\{\delta(x-x_0, y-y_0)\} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

f) Let

$$H(u,v) = CSFT\left\{\frac{1}{\sqrt{x^2+y^2}}\right\}$$

and let

$$CSFT\{f(x,y)\} = F(u,v)$$

$$CSFT\{w(x,y)\} = W(u,v)$$

where

$$w(x,y) = \mathcal{T}\{f(x,y)\}$$

then since this is a LSI system, we have that

$$W(u,v) = H(u,v)\,F(u,v)$$

So to recover $f(x,y)$ from $w(x,y)$ we do the following

1) $W(u,v) = CSFT\{w(x,y)\}$

2) $F(u,v) = \dfrac{W(u,v)}{H(u,v)}$

3) $f(x,y) = CSFT^{-1}\{F(u,v)\}$

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Problem 3.(25pt)

Consider an MRI system for which

$$\omega = LM$$

where ω is the resonance frequency in radians per second, L is the Lamar constant and M is the magnetic field; and let $s(t)$ denote the RF signal used to excite the target.

Furthermore, assume that the magnetic field can be controlled via three gradient coils, so that the magnetic field is given by

$$M(x,y,z) = M_0 + xG_x + yG_y + zG_z$$

where (x,y,z) are the spatial coordinates in 3-D.

a) (Slice select step.) Initially, the gradients are set to $G_x = 0$, $G_y = 0$, and $G_z = G_0 \neq 0$, and target is excited with the RF pulse

$$s(t) = e^{jLM_0 t}\mathrm{sinc}(t)\ .$$

For which values of (x,y,z) do the target atoms resonate? Justify your result.

b) Next the values of the x and y gradients are changed, but the z gradient remains fixed, so that $G_x = G_x(t)$, $G_y = G_y(t)$, and $G_z = G_0 \neq 0$. Let $\phi(t,x,y)$ be the phase of the signal radiated from a voxel of size (dx,dy,dz) at location (x,y), $z = 0$, and time t.

Compute an expression for $\phi(t,x,y)$ assuming that $\phi(0,x,y) = 0$.

c) Let $A(x,y)$ be the magnitude of the signal radiated from the voxel of size (dx,dy,dz) at location (x,y) and $z = 0$. Compute the total radiated signal from a slice of thickness dz at position $z = 0$.

a) $S(f) = CTFT\{e^{jLM_0 t}\,\mathrm{sinc}(t)\}$

$= \mathrm{rect}\left(f - \frac{LM_0}{2\pi}\right)$

frequencies range from

$$\frac{LM_0}{2\pi} - \frac{1}{2} \leq f \leq \frac{LM_0}{2\pi} + \frac{1}{2}$$

$$LM_0 - \pi \leq \omega \leq LM_0 + \pi$$

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since $\omega = L\, M(x,y,z)$

$$L M_0 - \pi \le L\, M(x,y,z) \le L M_0 + \pi$$

$$M_0 - \frac{\pi}{L} \le M(x,y,z) \le M_0 + \frac{\pi}{L}$$

$$M_0 - \frac{\pi}{L} \le M_0 + z G_0 \le M_0 + \frac{\pi}{L}$$

$$-\frac{\pi}{L G_0} \le z \le \frac{\pi}{L G_0}$$

b)

$$\phi(t) = \int_0^t L\left(M_0 + x G_x(\tau) + y G_y(\tau)\right) d\tau$$

$$= L M_0 t + x L \int_0^t G_x(\tau)\, d\tau$$

$$+ y L \int_0^t G_y(\tau)\, d\tau$$

define

$$K_x(t) = L \int_0^t G_x(\tau)\, d\tau$$

$$K_y(t) = L \int_0^t G_y(\tau)\, d\tau$$

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$$\phi(t) = L M_0 t + x k_x(t) + y k_y(t)$$

c) $$R(t) = \int_{\mathbb{R}^2} A(x,y)\, e^{j\phi(t,x,y)}\, dx\, dy$$

$$= \int_{\mathbb{R}^2} A(x,y)\, e^{jLM_0 t}\, e^{j(x k_x(t) + y k_y(t))}\, dx\, dy$$

$$= e^{jLM_0 t} \int_{\mathbb{R}^2} A(x,y)\, e^{j(x k_x(t) + y k_y(t))}\, dx\, dy$$