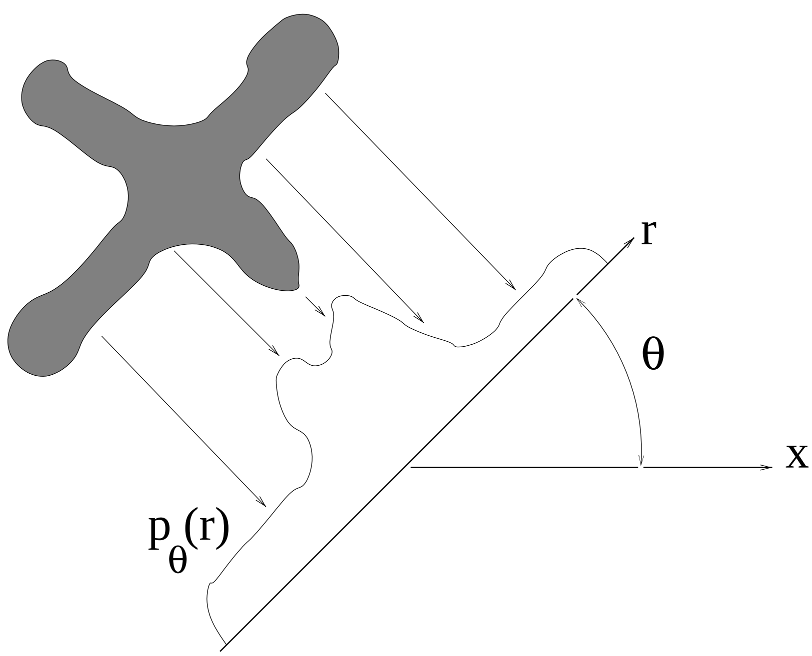
Application: Tomography

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.
  - Projections must be collected at every angle θ and displacement r.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.



- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

The Radon Transform

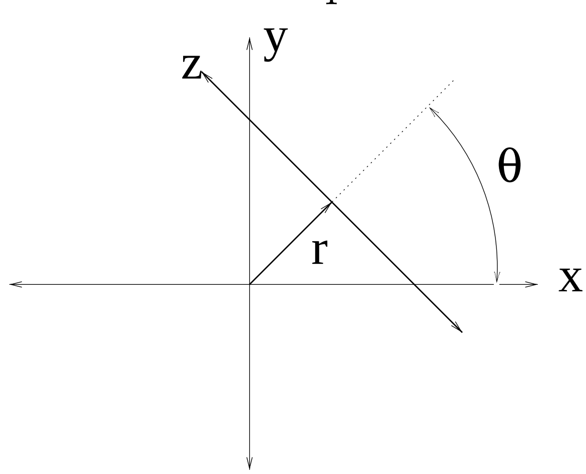
- The Radon transform is formed by projections of the image $f(x, y)$.
- Define the rotation matrix

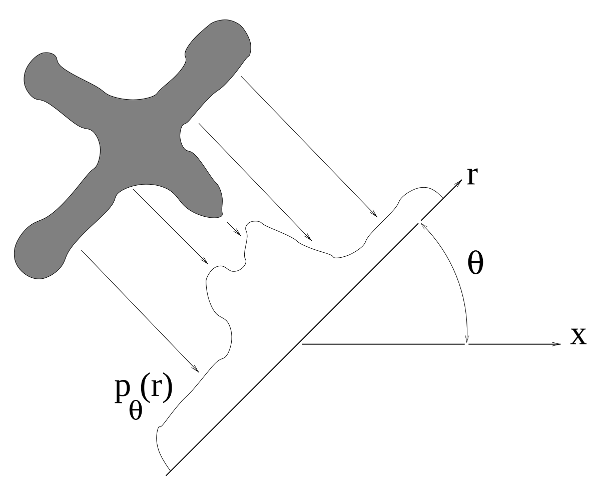
$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Then the radon transform is computed as

$$\begin{aligned} p_\theta(r) &= \int_{-\infty}^{\infty} f\left(\mathbf{A}_{-\theta}\begin{bmatrix} r \\ z \end{bmatrix}\right) dz \\ &= \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz \end{aligned}$$

- Geometric interpretation





Projection Geometry Projection at angle θ

The Fourier Slice Theorem

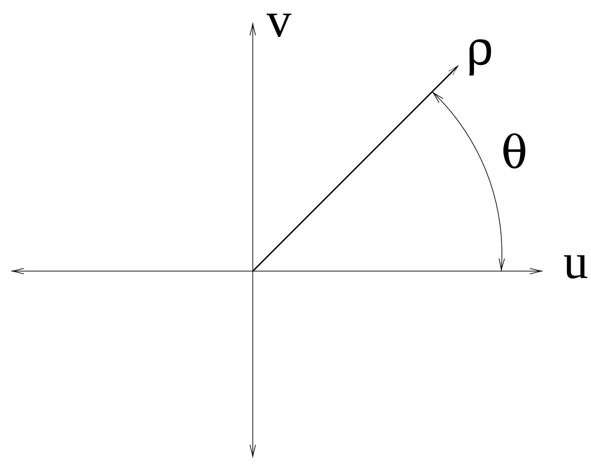
- Let

$$P_\theta(\rho) = CTFT\{p_\theta(r)\}$$
$$F(u,v) = CSFT\{f(x,y)\}$$

Then

$$P_\theta(\rho) = F(\rho\cos(\theta), \rho\sin(\theta))$$

- $P_\theta(\rho)$ is $F(u,v)$ in polar coordinates!



Proof of the Fourier Slice Theorem

- First let $\theta = 0$, then

$$p_0(r) \;=\; \int_{-\infty}^{\infty} f(r,y)\,dy$$

Then

$$\begin{aligned} P_0(\rho) \;&=\; \int_{-\infty}^{\infty} p_0(r)e^{-2\pi j r\rho}\,dr \\ &=\; \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(r,y)\,dy\right]e^{-2\pi j r\rho}\,dr \\ &=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,y)e^{-2\pi j(r\rho+y0)}\,dr\,dy \\ &=\; F(\rho,0) \end{aligned}$$

- By rotation property of CSFT, it must hold for any θ.

Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of $F(u,v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du\,dv = |\rho| d\theta\, d\rho$$

- This results in the expression

$$\begin{aligned} f(x,y) = &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{-2\pi j(xu+yv)}\, dudv \\ &= \int_{-\infty}^{\infty}\int_{0}^{\pi} P_\theta(\rho) e^{-2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))}\, |\rho| d\theta\, d\rho \\ &= \int_{0}^{\pi} \underbrace{\left[\int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{-2\pi j\rho(x\cos(\theta)+y\sin(\theta))}\, d\rho\right]}_{g_\theta(x\cos(\theta)+y\sin(\theta))} d\theta \end{aligned}$$

- Then $g(t)$ is given by

$$\begin{aligned} g_\theta(t) &= \int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{-2\pi j\rho t}\, d\rho \\ &= CTFT^{-1}\left\{|\rho| P_\theta(\rho)\right\} \\ &= h(t) * p_\theta(r) \end{aligned}$$

where $h(t) = CTFT^{-1}\left\{|\rho|\right\}$, and

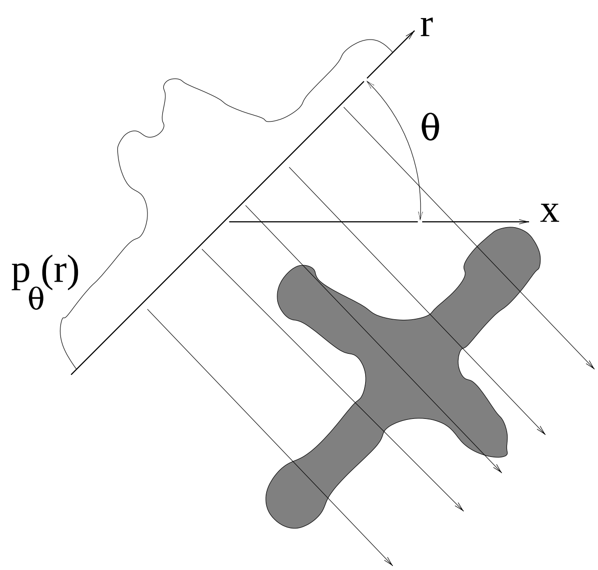
$$f(x,y) = \int_0^\pi g_\theta((x\cos(\theta) + y\sin(\theta))\, d\theta$$

Summary of CBP Algorithm

1. Compute projections $p_\theta(r)$.
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.
3. Back project filtered projections

$$f(x,y) = \int_0^\pi g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

- Back projection "smears" values of $g(t)$ back over image.



A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^\pi b_\theta\,(x, y)\,d\theta$$

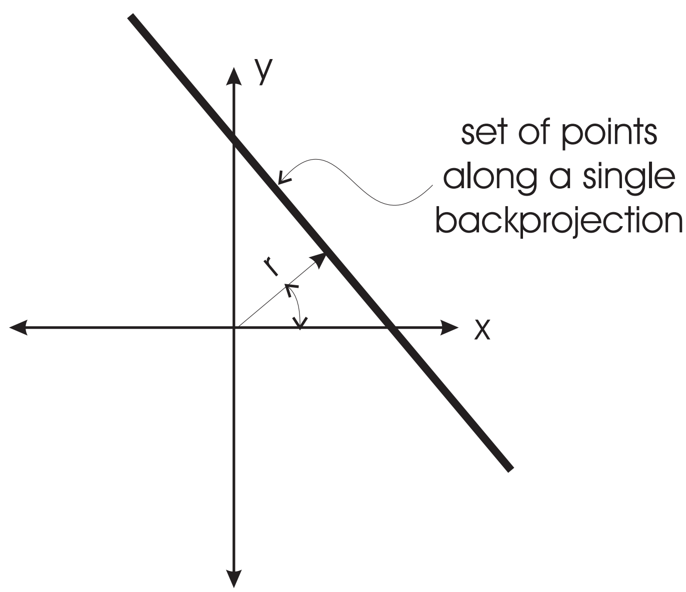
  where

$$b_\theta(x, y) = g_\theta(x\cos(\theta) + y\sin(\theta))$$

- Consider the set of points (x, y) such that

$$r = x\cos(\theta) + y\sin(\theta)$$

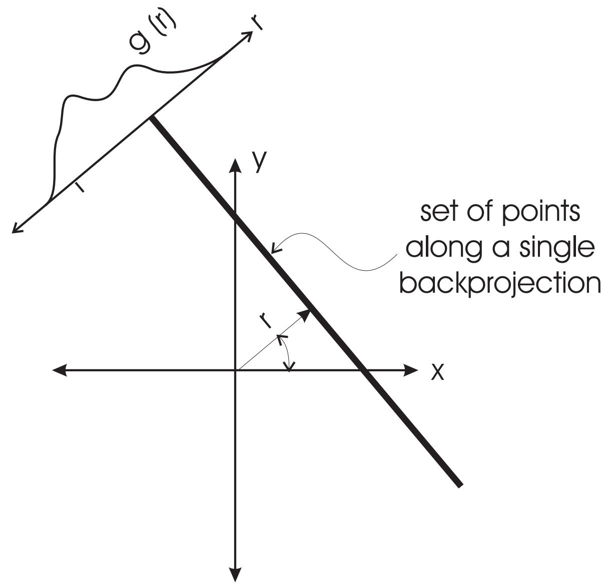
  This set looks like



- Along this line $b_\theta(x, y) = g_\theta(r)$.

Back Projection Continued

- For each angle θ back projection is constant along lines of angle θ and takes on value $g_\theta(r)$.



- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to π.

$$\begin{aligned} f(x,y) &= \int_0^\pi b_\theta(x,y)\,d\theta \\ &\approx \frac{\pi}{M}\sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x,y) \end{aligned}$$