Nonlinear Filtering

- Linear filters
 - Tend to blur edges and other image detail.
 - Perform poorly with non-Gaussian noise.
 - Result from Gaussian image and noise assumptions.
 - Images are not Gaussian.
- Nonlinear filter
 - Can preserve edges
 - Very effective at removing impulsive noise
 - Result from non-Gaussian image and noise assumptions.
 - Can be difficult to design.

Linear Filters

• Definition: A system y = T[x] is said to be linear if for all $\alpha, \beta \in IR$

$$\alpha y_1 + \beta y_2 = T[\alpha x_1 + \beta x_2]$$

where $y_1 = T[x_1]$ and $y_2 = T[x_2]$.

• Any filter of the form

$$y_s = \sum_r h_{s,r} x_r$$

Homogeneous Filters

• Definition: A filter y = T[x] is said to be homogeneous if for all $\alpha \in IR$

$$\alpha y = T[\alpha x]$$

- This is much weaker than linearity.
- Homogeneity is a natural condition for scale invariant systems.

Median Filter

- ullet Let W be a window with an odd number of points.
- Then the median filter is given by

$$y_s = \text{median} \{x_{s+r} : r \in W\}$$

- Is the median filter:
 - Linear?
 - Homogeneous?
- Consider the 1-D median filter with a 3-point window.

Median Filter: Optimization Viewpoint

• Consider the median filter

$$y_s = \text{median} \{x_{s+r} : r \in W\}$$

and consider the following functional.

$$F(\theta) \stackrel{\triangle}{=} \sum_{r \in W} |\theta - x_{s+r}|$$

• Then y_s solves the following optimization equation.

$$y_s = \arg\min_{\theta} F(\theta)$$

• Differentiating, we have

$$\frac{d}{d\theta}F(\theta) = \frac{d}{d\theta} \sum_{r \in W} |\theta - x_{s+r}|$$
$$= \sum_{r \in W} \operatorname{sign}(\theta - x_{s+r})$$
$$\triangleq f(\theta)$$

This expression only holds for $\theta \neq x_{s+r}$ for all $r \in W$.

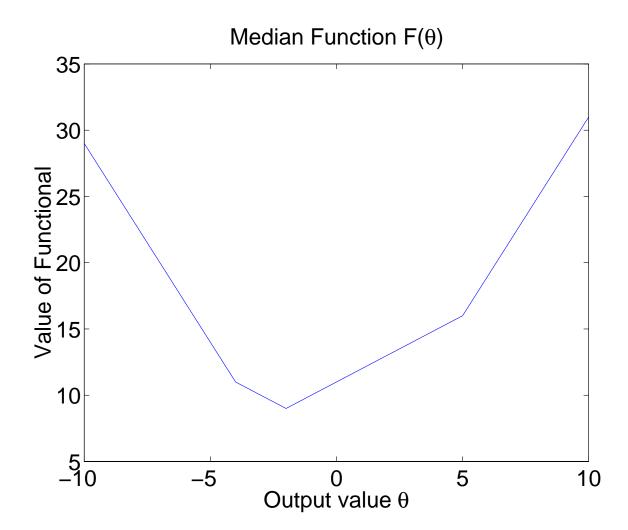
• So the solution falls at $\theta = x_{s*}$ such that

$$0 = \sum_{\substack{r \in W \\ r \neq (s*-s)}} \operatorname{sign}(\theta - x_{s+r})$$

Example: Median Filter Function

- Consider a 1-D median filter
 - Three point window of $W = \{-1, 0, 1\}$
 - Inputs [x(n-1), x(n), x(n+1)] = [-2, -4, 5].

$$F(\theta) = \sum_{k=-1}^{1} |\theta - x_{n+k}|$$

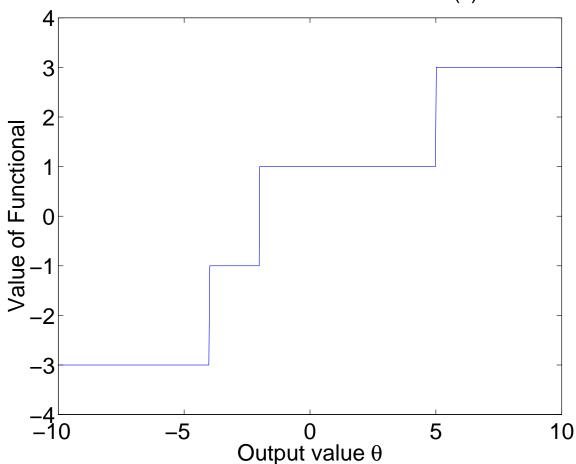


Example: Derivative of Median Filter Function

- Consider a 1-D median filter
 - Three point window of $W = \{-1, 0, 1\}$
 - Inputs [x(n-1), x(n), x(n+1)] = [-2, -4, 5].

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Derivative of Median Function $f(\theta)$

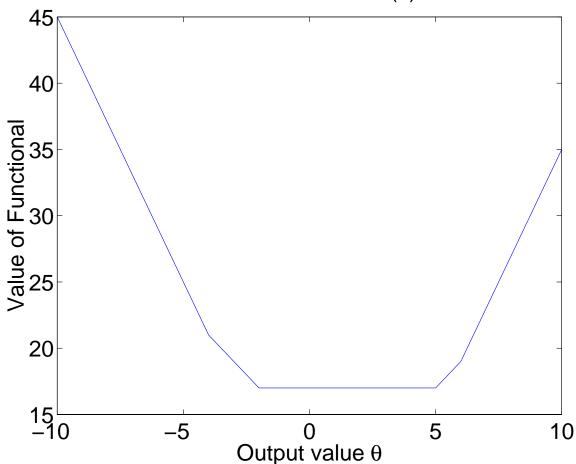


Problem with an Even Number of Points

- Consider a 1-D median filter
 - Four point window of $W = \{-1, 0, 1, 2\}$
 - Inputs [x(n-1), x(n), x(n+1), x(n+2)] = [-2, -4, 5, 6].
- Solution is not unique.

$$F(\theta) = \sum_{k=-1}^{2} |\theta - x_{n+k}|$$

Median Function $F(\theta)$



Weighted Median Filter

• Defined the functional

$$F(\theta) \stackrel{\triangle}{=} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

where a_r are weights assigned to each point in the window W.

• Weighted median is computed by

$$y_s = \arg\min_{\theta} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

• Differentiating, we have

$$\frac{d}{d\theta}F(\theta) = \frac{d}{d\theta} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

$$= \sum_{r \in W} a_r \operatorname{sign}(\theta - x_{s+r})$$

$$\stackrel{\triangle}{=} f(\theta)$$

This expression only holds for $\theta \neq x_r$ for all $r \in W$.

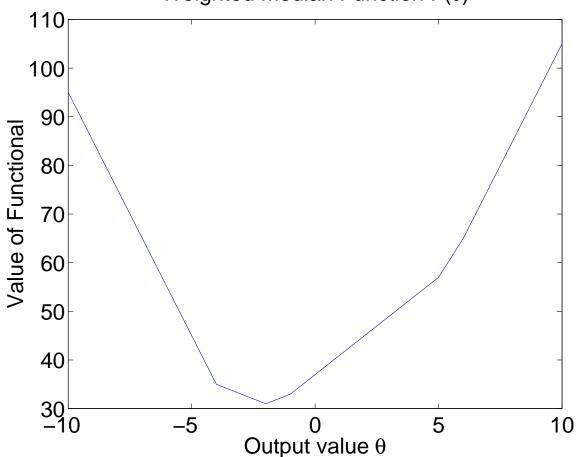
• Need to find s* such that $f(\theta)$ is "nearly" zero.

Example: Weighted Median Filter Function

- Consider a 1-D median filter
 - Five point window of $W = \{-2, -1, 0, 1, 2\}$
 - Inputs $[x(n-2), \dots, x(n+2)] = [6, -2, -4, 5, -1].$
 - Weights [a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1].

$$F(\theta) = \sum_{k=-1}^{1} a(k) |\theta - x_{n+k}|$$

Weighted Median Function $F(\theta)$

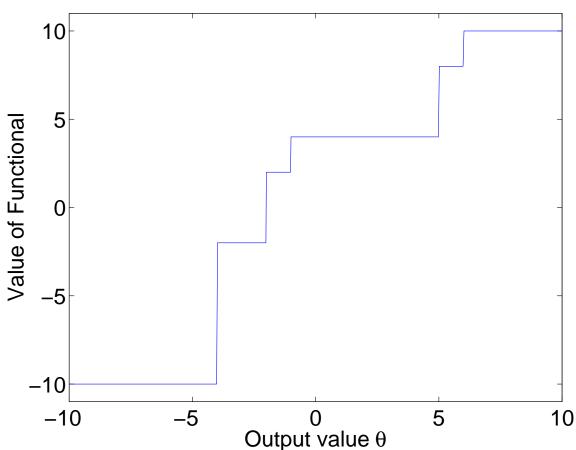


Example: Derivative of Median Filter Function

- Consider a 1-D median filter
 - Five point window of $W = \{-2, -1, 0, 1, 2\}$
 - Inputs $[x(n-2), \dots, x(n+2)] = [6, -2, -4, 5, -1].$
 - Weights [a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1].

$$f(\theta) = \sum_{k=-1}^{1} a(k) \operatorname{sign}(\theta - x_{n+k})$$

Derivative of Weighted Median Function $f(\theta)$



Computation of Weighted Median

- 1. Sort points in window.
 - Let $x_{(1)} < x_{(2)} < \cdots < x_{(p)}$ be the sorted values.
 - These values are known as order statistics.
 - Let $a_{(1)} < a_{(2)} < \cdots < a_{(p)}$ be the **corresponding** weights.
- 2. Find i* such that

$$\sum_{i=1}^{i*} a_{(i)} > \sum_{i=i*-1}^{p} a_{(i)}$$

and

$$\sum_{i=1}^{i*-1} a_{(i)} < \sum_{i=i*}^{p} a_{(i)}$$

3. Then the value $a_{(i*)}$ is the weighted median value.

Comments on Weighted Median Filter

- Weights may be adjusted to yeild the "best" filter.
- Largest and smallest values are ignored.
- Same as median filter for $a_r = 1$.