Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Note: The DTFT is periodic with period 2π.

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Therefore functions such as $\text{rect}(\omega)$ are not valid DTFT's.

Useful Discrete Time Functions

$$u(n) \triangleq \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\delta(n) \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\text{pulse}_N(n) \triangleq u(n) - u(n-N)$$

$$\text{psinc}_N(\omega) \triangleq \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$\text{prect}_{2\omega_o}(\omega) \triangleq \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 2\pi k}{2\omega_o}\right)$$

- $\text{psinc}_N(\omega)$ and $\text{prect}_{2\omega_o}(\omega)$ are periodic with period 2π.
- $\text{prect}_{2\omega_o}(\omega)$ has cut-off frequency ω_o.

Useful DTFT Transform Pairs

$$\delta(n) \overset{DTFT}{\Leftrightarrow} 1$$

$$1 \overset{DTFT}{\Leftrightarrow} 2\pi\delta(\omega)$$

$$\mathrm{pulse}_N(n) \overset{DTFT}{\Leftrightarrow} \mathrm{psinc}_N(\omega)e^{-j\omega\frac{N-1}{2}}$$

$$\mathrm{sinc}(Tn) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\mathrm{prect}_{2\pi T}(\omega)$$

$$\frac{\omega_o}{\pi}\mathrm{sinc}\left(\frac{\omega_0 n}{\pi}\right) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\mathrm{prect}_{2\omega_o}(\omega)$$

Discrete Space Fourier Transform (DSFT)

$$
\begin{aligned}
F(e^{j\mu}, e^{j\nu}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(m,n) e^{-j(\mu m + \nu n)} \\
f(m,n) &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(e^{j\mu}, e^{j\nu}) e^{j(\mu m + \nu n)} d\mu d\nu
\end{aligned}
$$

- Note: The DSFT is a 2-D periodic function with period 2π in in both the μ and ν dimensions.

$$F(e^{j(\mu+2\pi)}, e^{j(\nu+2\pi)}) = F(e^{j\mu}, e^{j\nu})$$

- Note: In matrix notation rows and columns are usually transposed.

$$f_{n,m} = f(m,n)$$

row $\rightarrow n$; column $\rightarrow m$

DSFT Properties Inherited from DTFT

- Some properties of the DSFT are directly inherited from the DTFT.

Property	Space Domain	DSFT
Linearity	$af(m,n)+bg(m,n)$	$aF(e^{j\mu},e^{j\nu})+bG(e^{j\mu},e^{j\nu})$
Conjugation	$f^*(m,n)$	$F^*(e^{-j\mu},e^{-j\nu})$
Shifting	$f(m-m_0,n-n_0)$	$e^{-j(\mu m_0+\nu y_0)}F(e^{j\mu},e^{j\nu})$
Modulation	$e^{-j2\pi(u_0m+v_0n)}f(m,n)$	$F(\mu-\mu_0,\nu-\nu_0)$
Convolution	$f(m,n)*g(m,n)$	$F(e^{j\mu},e^{j\nu})G(e^{j\mu},e^{j\nu})$
Multiplication	$f(m,n)g(m,n)$	$F(e^{j\mu},e^{j\nu})*G(e^{j\mu},e^{j\nu})$

- Inner product property

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)g^*(m,n)$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(e^{j\mu},e^{j\nu})G^*(e^{j\mu},e^{j\nu})d\mu d\nu$$

Properties Specific to DSFT

- Some properties are distinct to the DSFT.

Property	Space Domain	DSFT
Separability	$f(m)g(n)$	$F(e^{j\mu})G(e^{j\nu})$

- Notice that there are no scaling or rotation properties for the DSFT.

2-D Z-Transform Transform

- 1-D Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int_{C \in ROC} X(z) z^{n-1} dz$$

- 2-D Z-transform

$$F(z_1, z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n) z_1^{-m} z_2^{-n}$$

$$f(m,n) = \frac{1}{(2\pi j)^2} \int_{C^2 \in ROC} F(z_1, z_2) z_1^{m-1} z_2^{n-1} dz_1 dz_2$$

Relationship Between Fourier and Z-Transforms

- In 1-D, the DTFT is the 1-D Z-transform evaluated on the unit circle.

$$F(e^{j\omega}) = F(z)\big|_{z=e^{j\omega}}$$

- In 2-D the DSFT is the 2-D Z transform evaluated on the unit sphere.

$$F(e^{j\mu}, e^{j\nu}) = F(z_1, z_2)\big|_{z_1=e^{j\mu}, z_2=e^{j\nu}}$$