EE 301 Final Exam

December 13, Fall 2001

Name: Key

Instructions:

- Follow all instructions carefully!
- This is a 120 minute exam containing **six** problems totaling 150 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFS

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T} \end{aligned}$$

- CTFT

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega+a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$\begin{aligned} X_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \\ x[n] &= \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DTFT

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \end{aligned}$$

- DTFT pairs

$$a^n u[n] \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Sampling and Reconstruction

$$\begin{aligned} y[n] &= x(nT) \\ Y(\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y[k]\delta(t-kT) \\ S(\omega) &= Y(\omega T) \end{aligned}$$

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Problem 1.(25pt)

a) Prove that the DFT is an orthogonal transformation.

b) Derive Parseval's relation for the DFT.

c) Calculate X_k the N point DFT of $x[n] = e^{-j2\pi mn/N}$ for any integer m. Sketch the magnitude of X_k for $0 \leq k < N$.

d) Calculate X_k the N point DFT of $x[n] = \cos(-j2\pi mn/N)$ for any integer m. Sketch the magnitude of X_k for $0 \leq k < N$.

a) For $\phi_k = e^{jkn2\pi/N}$ we need to show that

$$\langle \phi_k, \phi_\ell \rangle = 0 \quad \text{for } k \neq \ell$$

$$\langle \phi_k, \phi_\ell \rangle = \sum_{n=0}^{N-1} \phi_k(n)\phi_\ell^*(n) = \sum_{n=0}^{N-1} e^{jkn2\pi/N} \cdot e^{-j\ell n 2\pi/N}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-\ell)n}$$

$$= \frac{1 - e^{j\frac{2\pi}{N}(k-\ell)N}}{1 - e^{j\frac{2\pi}{N}(k-\ell)}} = \frac{1 - \overbrace{e^{j2\pi(k-\ell)}}^{=1 \text{ for } k\neq\ell}}{1 - e^{j\frac{2\pi}{N}(k-\ell)}} = 0$$

For $k \neq \ell \Rightarrow \langle \phi_k, \phi_\ell \rangle = 0 \Rightarrow$ orthogonal //

b)

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn} \qquad (*)$$

$$x^*(n) = \left[\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn}\right]^* = \sum_{k=0}^{N-1} X_k^* e^{-j\frac{2\pi}{N}kn} \qquad (**)$$

Let $k = m$ in $**$ and multiply them $\longrightarrow$

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$$\underbrace{x(n)\cdot x^*(n)}_{=|x(n)|^2} = \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} X_k X_m^* \, e^{j\frac{2\pi}{N}kn}\, e^{-j\frac{2\pi}{N}mn}$$

Take the summation of both sides w.r.t. n :

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} X_k X_m^* \underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} \cdot e^{-j\frac{2\pi}{N}mn}}_{=\begin{cases} 0 & k\neq m \\ N & k=m\end{cases}}$$

So $\displaystyle\sum_{n=0}^{N-1} |x(n)|^2 = N\sum_{k=0}^{N-1} |X_k|^2$ //

c)

$$X_k = (1/N)\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}mn} \cdot e^{j\frac{2\pi}{N}kn}$$

$$= 1/N \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n(k-m)}$$

From part (a) we know that if $k\neq m$ this is zero. If $k=m$ we get $X_k = 1/N \sum_{n=0}^{N-1} = 1$. So

$$X_k = \left\{\begin{matrix} 0 & 0\leq k<m \\ 1 & k=m \\ 0 & m<k<N \end{matrix}\right\} = \delta(k-m)$$

But since X_k is periodic with period N $\longrightarrow$

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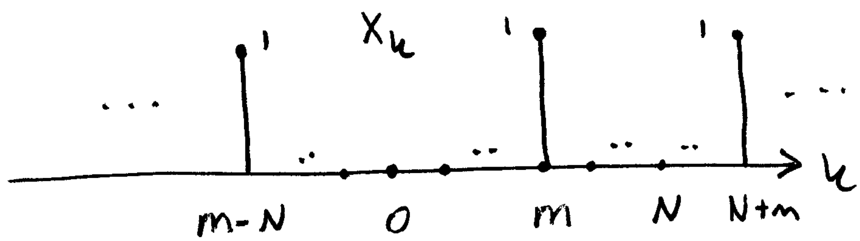
$$X_k = \sum_{l=-\infty}^{\infty} \delta(k-m-lN)$$ //



d) $$\cos\left(\frac{2\pi m n}{N}\right) = \frac{e^{j\frac{2\pi}{N}mn}}{2} + \frac{e^{-j\frac{2\pi}{N}mn}}{2}$$

Using the result from part (c) :

$$X_k = \frac{1}{2}\left[\sum_{l=-\infty}^{\infty} \delta(k-m-lN) + \delta(k+m-lN)\right]$$ //

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Problem 2.(25pt)

Consider a discrete-time system $y[n] = T[x[n]]$ where the input $x[n]$ and the output $y[n]$ obey the following difference equation

$$y[n] = ay[n-1] + x[n]$$

where $|a| < 1$.

a) Prove that this system is either linear or nonlinear.

b) Prove that the system is either time varying or time invariant.

c) Calculate the frequency response of the system $H(\omega)$, and use this result to calculate the impulse response $h[n]$.

d) Calculate the output $y[n]$ when the input is $x[n] = b^n u[n]$ where $a \neq b$ and $|b| < 1$.

a) Let $\alpha, \beta \in \mathbb{R}$ and $x_1(n)$ and $x_2(n)$ be any two functions. Then:

$$T[x_1(n)] = y_1(n) = ay_1(n-1) + x_1(n) \qquad (1)$$

$$T[x_2(n)] = y_2(n) = ay_2(n-1) + x_2(n) \qquad (2)$$

Multiply (1) by α, (2) by β and add.

$$\alpha y_1(n) + \beta y_2(n) = \alpha a y_1(n-1) + \alpha x_1(n) + \beta a y_2(n-1) + \beta x_2(n)$$

$$= a[\alpha y_1(n-1) + \beta y_2(n-1)] + \alpha x_1(n) + \beta x_2(n)$$

Let $x_3(n) = \alpha x_1(n) + \beta x_2(n)$

$y_3(n) = \alpha y_1(n) + \beta y_2(n)$

$\Leftrightarrow y_3(n) = a y_3(n) + x_3(n) \Leftrightarrow y_3(n) = T[x_3(n)]$

$\Leftrightarrow \alpha y_1(n) + \beta y_2(n) = T[\alpha x_1(n) + \beta x_2(n)] \Rightarrow$ Linear//

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b) Let n_0 be an integer and define

$$x'(n) = x(n-n_0) \text{ and } y'(n) = y(n-n_0).$$

The system is TI if $y'(n) = T[x'(n)]$.

We know that: $y(n) = ay(n-1) + x(n)$

$$\Leftrightarrow \quad y(n-n_0) = ay(n-n_0-1) + x(n-n_0)$$

$$\Leftrightarrow \quad y'(n) = ay'(n) + x'(n)$$

$$\Leftrightarrow \quad y'(n) = T[x'(n)] \Leftrightarrow \text{system is TI} \;//$$

c) Take the DTFT of both sides:

$$Y(\omega) = ae^{-j\omega} Y(\omega) + X(\omega)$$

$$\Leftrightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1-ae^{-j\omega}} \leftrightarrow h(n) = a^n u(n) \;//$$

d) $x(n) = b^n u(n) \leftrightarrow X(\omega) = \dfrac{1}{1-be^{-j\omega}}$

So $$Y(\omega) = H(\omega)Y(\omega) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

$$= \frac{A}{1-ae^{-j\omega}} + \frac{B}{1-be^{-j\omega}}$$

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Using the cover-up method, we obtain:

$$A = \frac{a}{a-b}, \qquad B = -\frac{b}{a-b}$$

So $$y(n) = \left[\left(\frac{a}{a-b}\right)a^n - \left(\frac{b}{a-b}\right)b^n\right]u(n)$$ //

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Problem 3.(25pt)

Consider the discrete time LTI system $y[n] = T[x[n]]$ with real valued input and output and frequency response $H(\omega) = A(\omega)e^{j\phi(\omega)}$ where $A(\omega)$ and $\phi(\omega)$ are real valued functions of ω.

a) State whether $A(\omega)$ and $\phi(\omega)$ are symmetric or antisymmetric functions of ω.

b) Calculate the output $y[n]$ when the input is $x[n] = e^{j\omega_0 n}$. Your result should be in terms of the functions $A(\omega)$ and $\phi(\omega)$.

c) Calculate the output $y[n]$ when the input is $x[n] = \cos(\omega_0 n)$. Your result should be in terms of the functions $A(\omega)$ and $\phi(\omega)$.

d) Calculate the output $y[n]$ when the input is $x[n] = a + b(-1)^n + \sin(\omega_0 n)$. Your result should be in terms of the functions $A(\omega)$ and $\phi(\omega)$.

a)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X^*(\omega) = \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}\right]^* = \sum_{n=-\infty}^{\infty} x^*(n) e^{j\omega n}$$

If $x(n)$ is always real, then $x^*(n) = x(n)$ and we get:

$$X^*(-\omega) = X(\omega) \quad \text{if } x(n) \text{ real.} \quad (*)$$

Let $X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$. Using $(*)$ we see that:

$$X^*(-\omega) = |X(-\omega)| e^{-j\angle X(-\omega)} = X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

$\Leftrightarrow$ $|X(-\omega)| = |X(\omega)| \Rightarrow$ magnitude is a symmetric func.

$-\angle X(-\omega) = \angle X(\omega) \rightarrow$ phase is an asymmetric func.

The same argument holds for $Y(\omega)$ since we know that $y(n)$ is always real. $\longrightarrow$

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$$H(\omega) = A(\omega) e^{J\phi(\omega)} = \frac{Y(\omega)}{X(\omega)} = \frac{|Y(\omega)|}{|X(\omega)|} \frac{e^{J\measuredangle Y(\omega)}}{e^{J\measuredangle X(\omega)}}$$

$$\Leftrightarrow \quad A(\omega) = \frac{|Y(\omega)|}{|X(\omega)|} \rightarrow \text{symmetric} \;//$$

$$\phi(\omega) = \measuredangle Y(\omega) - \measuredangle X(\omega) \rightarrow \text{antisymmetric} \;//$$

b) $\quad y(n) = x(n) * h(n) \Longleftrightarrow Y(\omega) = X(\omega) H(\omega)$

$$x(n) = e^{Jw_0 n} \Longleftrightarrow X(\omega) = \sum_{l=-\infty}^{\infty} \delta(\omega - w_0 - 2\pi l)$$

So $\quad Y(\omega) = H(\omega) \sum_{l=-\infty}^{\infty} \delta(\omega - w_0 - 2\pi l)$ $\downarrow$ sifting property.

$$= \sum_{l=-\infty}^{\infty} H(w_0) \delta(\omega - w_0 - 2\pi l)$$

$$\Leftrightarrow \quad y(n) = H(w_0) e^{Jw_0 n} = e^{Jw_0 n} \left(A(w_0) e^{J\phi(w_0)} \right) \;//$$

c) $\quad \cos w_0 n = \frac{1}{2} e^{Jw_0 n} + \frac{1}{2} e^{-Jw_0 n}$

So $\quad y(n) = T[\cos w_0 n] = T\left[\frac{1}{2} e^{Jw_0 n} + \frac{1}{2} e^{-Jw_0 n} \right]$

$$= \frac{1}{2} e^{Jw_0 n} A(w_0) e^{J\phi(w_0)} + \frac{1}{2} e^{-Jw_0 n} A(-w_0) e^{J\phi(-w_0)}$$

$A(\omega)$ is symmetric $\Rightarrow A(-w_0) = A(w_0)$

$\phi(\omega)$ is antisymmetric $\Rightarrow \phi(-w_0) = -\phi(w_0)$.

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$$\text{So } y(n) = \frac{1}{2} A(w_0) e^{J(w_0 n + \phi(w_0))} + \frac{1}{2} A(w_0) e^{-J(w_0 n + \phi(w_0))}$$

$$= A(w_0) \cos (w_0 n + \phi(w_0)) \;//$$

d) $b(-1)^n = b e^{J\pi n}$, $a = a e^{J0n}$, and

$$\sin w_0 n = \frac{1}{2J} e^{Jw_0 n} - \frac{1}{2J} e^{-Jw_0 n}$$

$$\text{So } y(n) = T\left[a e^{J0n} + b e^{J\pi n} + \frac{1}{2J} e^{Jw_0 n} - \frac{1}{2J} e^{-Jw_0 n}\right]$$

$$= aH(0) + b e^{J\pi n} H(\pi) + \frac{1}{2J} e^{Jw_0 n} H(w_0) - \frac{1}{2J} e^{-Jw_0 n} H(-w_0)$$

$$H(0) = A(0) e^{J\phi(0)} = A(0)$$

$$H(w_0) = A(w_0) e^{J\phi(w_0)}, \quad H(-w_0) = A(w_0) e^{-J\phi(w_0)}$$

$$\Leftrightarrow y(n) = a A(0) + b(-1)^n A(\pi) e^{J\phi(\pi)} + \frac{A(w_0)}{2J} e^{J(w_0 n + \phi(w_0))} - \frac{A(w_0)}{2J} e^{-J(w_0 n + \phi(w_0))}$$

$$\Leftrightarrow y(n) = a A(0) + b(-1)^n A(\pi) e^{J\phi(\pi)} + A(w_0) \sin (w_0 + \phi(w_0)) \;//$$

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Problem 4.(25pt)

The input to a radar receiver is a discrete time signal $x[n] = e^{-j\omega_0 n}$. Define the "window" function of length N as $w[n] = u[n] - u[n-N]$. Then the radar signal processor "windows" input to form the received signal $r[n]$.

$$r[n] = w[n]\, x[n]$$

a) Calculate $X(\omega)$ the DTFT of $x[n]$ and sketch a plot of its magnitude. Make sure to label all relevant quantities on the plot.

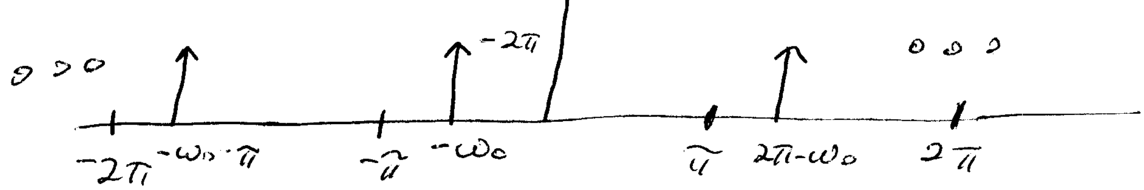
b) Calculate $W(\omega)$ the DTFT of $w[n]$ and sketch a plot of its magnitude. Make sure to label all relevant quantities on the plot.

c) Calculate $R(\omega)$ the DTFT of $r[n]$ and sketch a plot of its magnitude. Make sure to label all relevant quantities on the plot.

d) Calculate R_k the DFT of $r[n]$ and sketch a plot of its magnitude. How does your plot depend on the value of ω_0?

a) $e^{-j\omega_0 n} \overset{DTFT}{\Longleftrightarrow} 2\pi\,\delta(\omega+\omega_0)$ for $|\omega| < \pi$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi k)$$



b) $$W(\omega) = \sum_{n=-\infty}^{\infty} w[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

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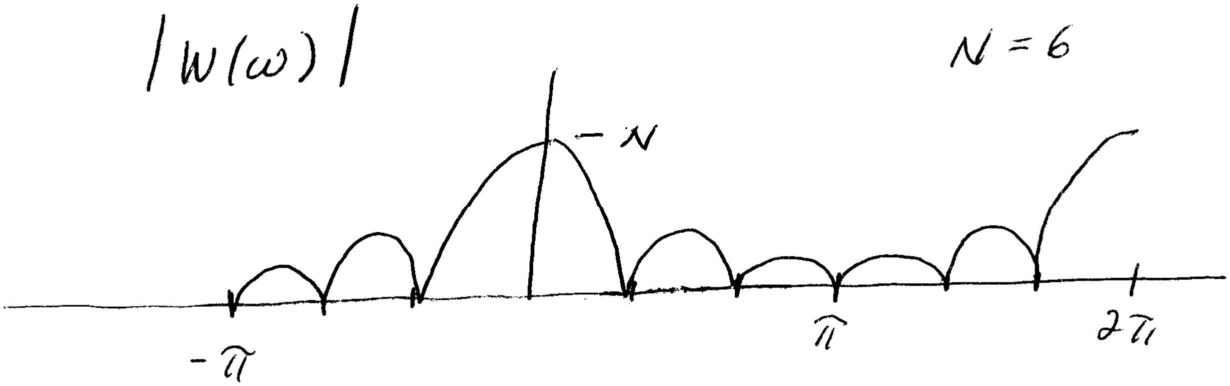
$$= \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= e^{-j\omega\frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$|W(\omega)|$



c) $R(\omega) = \mathrm{DTFT}\{ w[n] e^{-j\omega_0 n} \}$

$$= W(\omega + \omega_0)$$

$$= e^{-j(\omega+\omega_0)\frac{N-1}{2}} \frac{\sin((\omega+\omega_0)N/2)}{\sin((\omega+\omega_0)/2)}$$

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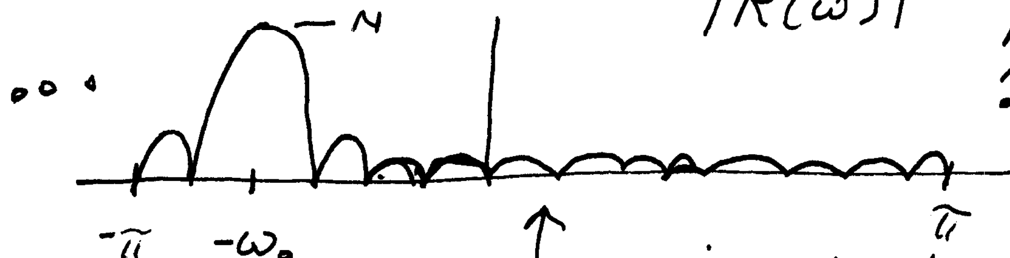
$|R(\omega)|$

repeats periodicly ...



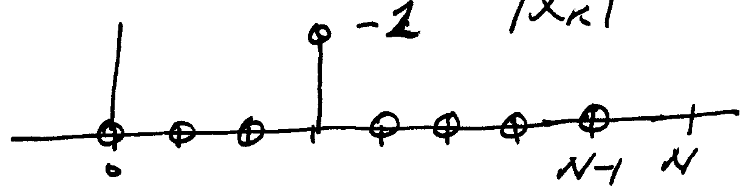
contains a total of $N-1$ sidelobes between $-\pi$ an π

d) $R_K = \frac{1}{N}\sum_{n=0}^{N-1} r[n]\, e^{-j2\pi \frac{kn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j(2\pi \frac{k}{N} + \omega_0) n}$

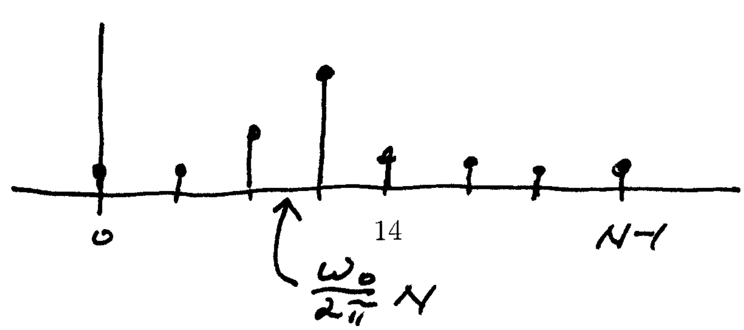
$$= \frac{1}{N}\,\frac{1 - e^{-j(2\pi \frac{k}{N} + \omega_0) N}}{1 - e^{-j(2\pi \frac{k}{N} + \omega_0)}}$$

$$= \frac{1}{N} e^{-j(2\pi \frac{k}{N} + \omega_0)\frac{N-1}{2}} \quad \frac{\sin\left((2\pi \frac{k}{N} + \omega_0) N\right)}{\sin\left(2\pi \frac{k}{N} + \omega_0\right)}$$

When $\omega_0 = 2\pi \frac{m}{N}$



When $\omega_0 \neq 2\pi \frac{m}{N}$ for ANY m then



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Problem 5.(25pt) Consider system designed to sample a signal $x(t)$ and reconstruct a signal $z(t)$. Assume that the signal $x(t)$ is band limited to frequencies $|\omega| < \omega_s/2$ where $\omega_s = 2\pi/T$. The system is governed by the following set of equations.

$$\begin{aligned} y[n] &= x(nT) \\ s(t) &= \sum_{k=-\infty}^{\infty} y[k]\delta(t-kT) \\ z(t) &= s(t) * \mathrm{sinc}(t/T) \end{aligned}$$

Furthermore, let $X(\omega)$ be the CTFT of $x(t)$, let $Y(\omega)$ be the DTFT of $y[n]$, let $S(\omega)$ be the CTFT of $s(t)$, let $Z(\omega)$ be the CTFT of $z(t)$, and let $H(\omega)$ be the CTFT of $h(t)$.
(For all sketches in this section, label all relevant quantities, and make sure that all your plots are consistent with each other.)

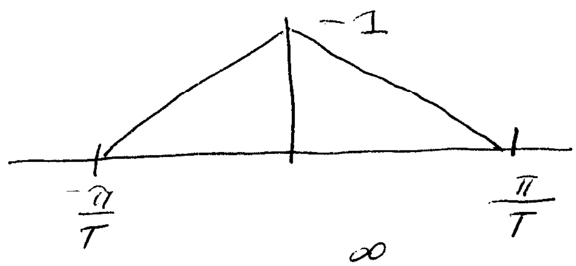
a) Sketch the magnitude of a typical function $X(\omega)$.

b) Calculate $Y(\omega)$ in terms of the function $X(\omega)$ and sketch the magnitude of $Y(\omega)$.

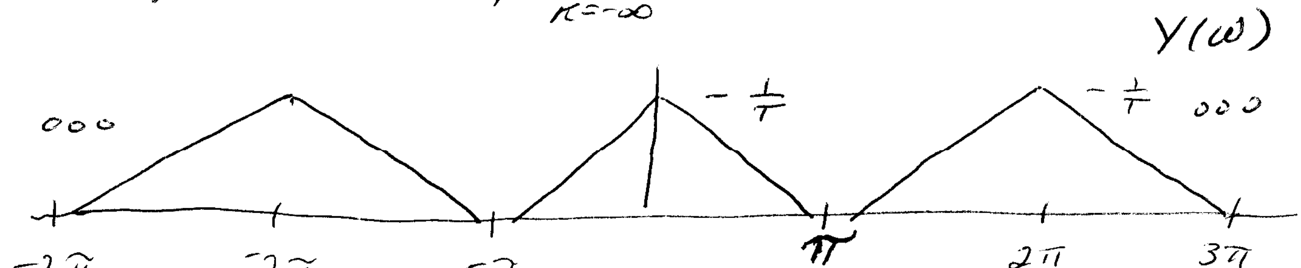
c) Calculate $S(\omega)$ in terms of the function $X(\omega)$ and sketch the magnitude of $S(\omega)$.

d) Calculate $Z(\omega)$ in terms of the function $X(\omega)$ and sketch the magnitude of $Z(\omega)$.

a)



b) $$Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{T}\right)$$



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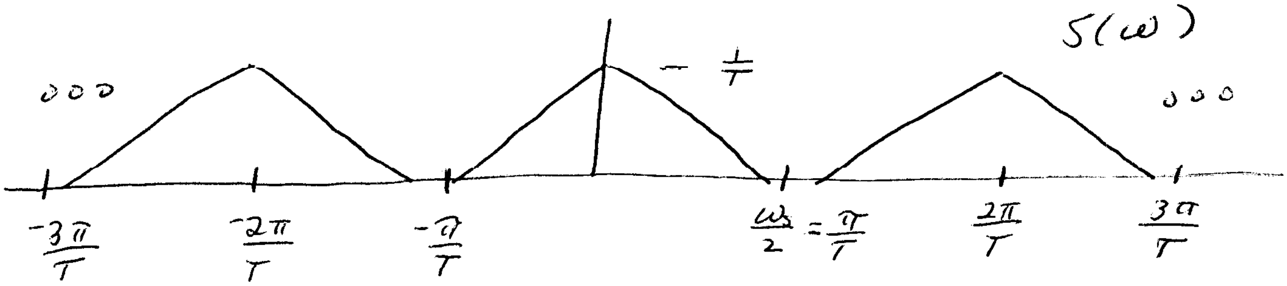
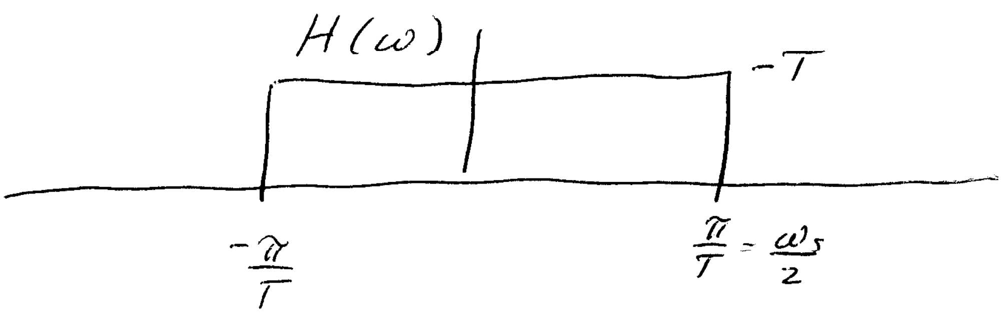
c) $S(\omega) = Y(\omega T)$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}k\right)$$



d) $h(t) = \mathrm{sinc}(t/T) \overset{CTFT}{\Longleftrightarrow} T\,\mathrm{rect}(\omega T/2\pi)$

$$= T\,\mathrm{rect}(\omega/\omega_s)$$



$$Z(\omega) = S(\omega)\cdot H(\omega)$$

$$= \left(\frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi k}{T}\right)\right) T\,\mathrm{rect}(\omega/\omega_s)$$

$$= \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}k\right)\mathrm{rect}(\omega/\omega_s)$$

$$= X(\omega)$$

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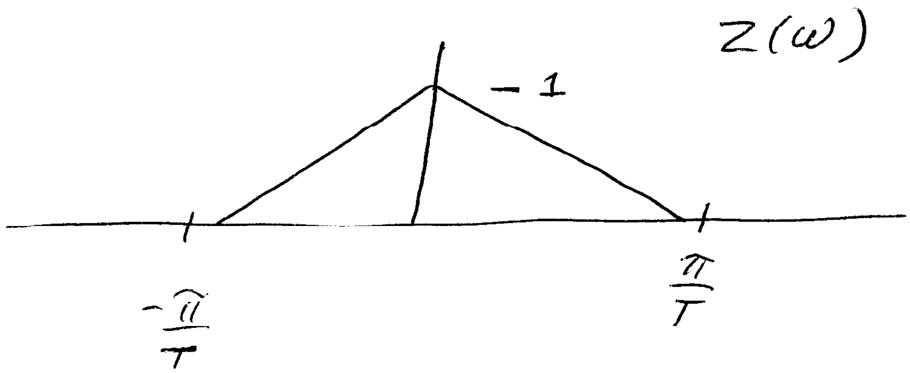

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Problem 6.(25pt) Consider a digital signal processing system with input $x(t)$ and output $z(t)$ where the input signal $x(t)$ is band limited to frequencies $|\omega| < \omega_s/2$ for $\omega_s = 2\pi/T$. The system is governed by the following set of equations.

$$\begin{aligned} y[n] &= x(nT) \\ f[n] &= y[n] * g[n] \\ s(t) &= \sum_{k=-\infty}^{\infty} f[k]\delta(t-kT) \\ z(t) &= s(t) * h(t) \end{aligned}$$

Furthermore, let $X(\omega)$ be the CTFT of $x(t)$, let $Y(\omega)$ be the DTFT of $y[n]$, let $G(\omega)$ be the DTFT of $g[n]$, let $F(\omega)$ be the DTFT of $f[n]$, let $S(\omega)$ be the CTFT of $s(t)$, let $Z(\omega)$ be the CTFT of $z(t)$, and let $H(\omega)$ be the CTFT of $h(t)$.

a) Calculate the $Z(\omega)$ in terms of the functions $X(\omega)$, $G(\omega)$, and $H(\omega)$.

b) Consider a digital recording system which we will refer to as DRA. The system DRA uses no digital filter (i.e. $f[n] = y[n]$) and reconstructs the signal using a standard digital-to-analog converter (D/A converter) running at a frequency of $1/T$. Real D/A converters can be modeled as generating square pulses of width T, so $h(t) = \text{rect}(t/T)$.

Calculate the $Z(\omega)$ in terms of the function $X(\omega)$ for the system DRA.

c) Is the system DRA a LTI system? If so, why? If not, why not?

d) A second digital recording system is designed which we will refer to as DRB. The system DRB also uses a real D/A converter running at frequency $1/T$, but the filter $G(\omega)$ is chosen so that $z(t) = x(t)$.

What is the value of $G(\omega)$?

a)
$$Y(\omega) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\left(\frac{\omega - 2\pi K}{T}\right)$$

$$F(\omega) = G(\omega)\, Y(\omega)$$

$$= \frac{G(\omega)}{T} \sum_{K=-\infty}^{\infty} X\left(\frac{\omega - 2\pi K}{T}\right)$$

$$S(\omega) = F(\omega T)$$

$$= \frac{G(\omega T)}{T} \sum_{K=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T} K\right)$$

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$$Z(\ ) = \frac{H(\omega)G(\omega T)}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}k\right)$$

b) For DRA we have

$$G(\omega) = 1$$

$$H(\omega) = CTFT\{ rect(t/T) \}$$

$$rect(t/T) \overset{CTFT}{\Longleftrightarrow} T\,sinc(\omega/(2\pi/T))$$

$$= T\,sinc(\omega/\omega_s)$$

$$H(\omega) = T\,sinc(\omega/\omega_s)$$

So we have that

$$(*) \quad Z(\omega) = \frac{T\,sinc(\omega/\omega_s)}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}k\right)$$

c) The system is <u>linear</u> because the relationship $(*)$ is a linear function of the function $X(\cdot)$.

However, the system is <u>NOT</u> LTI because the equation $(*)$ does not have the form $Z(\omega) = H(\omega) \cdot X(\omega)$

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for some function $\mathcal{H}(\omega)$.

Therefore, the system must be <u>time varying</u>.

d) For DRB, we have

$$H(\omega) = T \operatorname{sinc}(\omega/\omega_s)$$

So

$$Z(\omega) = \frac{G(\omega T)\, T \operatorname{sinc}(\omega/\omega_s)}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}k\right)$$

$$(**) \quad Z(\omega) = G(\omega T) \operatorname{sinc}(\omega/\omega_s) \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}k\right)$$

The original exam contained an error, so no value of $G(\omega)$ will allow

$$Z(\omega) = X(\omega)$$

due to the form of (**). //

The corrected exam states the the output of the D/A converter is low pass filtered to ~~f~~requency $\frac{\omega_s}{2} = \frac{\pi}{T}$.

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For this case,

$$Z(\omega) = G(\omega T)\,\mathrm{sinc}(\omega/\omega_s)\,X(\omega)$$

So we need for $|\omega| < \pi$

$$G(\omega T)\,\mathrm{sinc}(\omega/\omega_s) = 1$$

$$G(\omega T) = \frac{1}{\mathrm{sinc}(\omega/\omega_s)}$$

$$G(\omega) = \frac{1}{\mathrm{sinc}(\omega/(T\omega_s))}$$

$$= \frac{1}{\mathrm{sinc}(\omega/2\pi)} \quad \text{for } |\omega| < \pi$$