2-D Discrete Space Random Processes

- Notation
  - X_s is a pixel at position $s = (s_1, s_2) \in \mathcal{Z}^2$
  - S denotes the set of 2-D Lattice points where $S \subset \mathcal{Z}^2$
- Definitions
  - Mean $\mu_s = E[X_s]$
  - Autocorrelation $R_{sr} = E[X_s X_r]$
  - Autocovariance $C_{sr} = E[(X_s - \mu_s)(X_r - \mu_r)]$
  - A process is said to be **second order** if $E[X_s]$ and $E[X_s X_r]$ exist for all $s \in S$ and $r \in S$.
  - A second order random process is said to be **wide sense stationary** if for all $s \in \mathcal{Z}^2$

$$
\begin{aligned}
\mu_s &= \mu_{(0,0)} \\
C_{r,r+s} &= C_{(0,0),s}
\end{aligned}
$$

2-D Power Spectral Density

Let X_s be a zero mean wide sense stationary random process.

Define

$$X_N(e^{j\mu}, e^{j\nu}) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} X_{(m,n)} e^{j(m\mu n\nu)}$$

- Then the power spectrum (i.e. energy spectrum per unit sample) is

$$\frac{1}{(2N+1)^2} \left| X_N(e^{j\mu}, e^{j\nu}) \right|^2$$

The follow limit does not converge!!

$$\lim_{N \to \infty} \frac{1}{(2N+1)^2} \left| X_N(e^{j\mu}, e^{j\nu}) \right|^2$$

Intuition - The spectral estimate remains noisy as the window size increases i.e. the signal does not decay.

- Definition of **Power Spectral Density**

$$S_x(e^{j\mu}, e^{j\nu}) \triangleq \lim_{N \to \infty} \frac{1}{(2N+1)^2} E\left[\left| X_N(e^{j\mu}, e^{j\nu}) \right|^2 \right]$$

Weiner-Khintchine Theorem

- For a wide sense stationary random process, the power spectral density is the Fourier transform of the autocorrelation

$$S_x(e^{j\mu}, e^{j\nu}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m,n) e^{-j(m\mu + n\nu)}$$

where

$$R(m,n) = E[X_{(0,0)} X_{(m,n)}]$$

Filtered Random Processes

- Consider the 2-D linear system

$$Y(m,n) = h(m,n) * X(m,n)$$

where $X(m,n)$ is a 2-D wide sense stationary random process.

- It may be easily shown that

$$\begin{aligned} R_y(m,n) &= R_x(m,n) * h(m,n) * h(-m,-n) \\ S_y(e^{j\mu}, e^{j\nu}) &= \left| H(e^{j\mu}, e^{j\nu}) \right|^2 S_x(e^{j\mu}, e^{j\nu}) \end{aligned}$$

Filtering White Noise

- Let $X(m,n)$ be independent identically distributed (i.i.d.) random variables with distribution Gaussian $N(0,1)$, and let $Y(m,n) = h(m,n) * X(m,n)$.

- $X(m,n)$ is wide sense stationary with

$$
\begin{aligned}
\mu(m,n) &= 0 \\
R_x(k,l) &= E[X(0,0)X(k,l)] \\
&= \delta(k,l) \\
S_x(e^{j\mu}, e^{j\nu}) &= \mathcal{F}\{R_x(k,l)\} \\
&= 1
\end{aligned}
$$

- $Y(m,n)$ is wide sense stationary with

$$
\begin{aligned}
S_y(e^{j\mu}, e^{j\nu}) &= \left|H(e^{j\mu}, e^{j\nu})\right|^2 \cdot 1 \\
R_y(k,l) &= h(m,n) * h(-m,-n) \\
&= \sum_{m=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(m,n)h(m+k, n+l)
\end{aligned}
$$

- $R_y(k,l)$ is the autocorrelation of $h(m,n)$ with itself.

Causal Prediction

- Let Y_s by a 2-D wide sense stationary Gaussian random process.
- Define
  - The past values are $Y_{<s} = \{Y_r : r < s\}$.
  - The minimum mean squared error (MMSE) predictor of Y_s given the past is

$$\hat{Y}_s = E[Y_s | Y_{<s}]$$

  - The prediction error is $X_s = Y_s - \hat{Y}_s$.
- Fact 1: for $r < s$

$$\begin{aligned} E[X_s X_r] &= E[E[X_s X_r | Y_{<s}]] \\ &= E[E[(Y_s - \hat{Y}_s)(Y_r - \hat{Y}_r) | Y_{<s}]] \\ &= E[E[(Y_s - \hat{Y}_s) | Y_{<s}](Y_r - \hat{Y}_r)] \\ &= E[(E[Y_s | Y_{<s}] - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\ &= E[(\hat{Y}_s - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\ &= E[0(Y_r - \hat{Y}_r)] = 0 \end{aligned}$$

- Fact 2: $\sigma^2 \triangleq E[X_s^2]$ is the prediction variance
- Fact 3: The causal predictor must be time invariant and linear $\Rightarrow \hat{Y}_s = \sum_{r>(0,0)} h_r Y_{s-r}$

Minimum Mean Squared Error Linear Prediction

- Let Y_s by a 2-D wide sense stationary Gaussian random process.
- Let h_s be a MMSE linear predictor for Y_s.
- Then

$$
\begin{aligned}
&X_s = Y_s - h_s * Y_s \\
&X(e^{j\mu}, e^{j\nu}) = (1 - H(e^{j\mu}, e^{j\nu}))Y(e^{j\mu}, e^{j\nu}) \\
&R_x(s) = \sigma^2 \delta(s) \\
&S_x(e^{j\mu}, e^{j\nu}) = \sigma^2
\end{aligned}
$$

- If h_s is FIR, then Y_s is known as an autoregressive (AR) random process.

Autoregressive Processes

- Let h_s be a MMSE linear predictor for Y_s.
- Then $X(e^{j\mu}, e^{j\nu}) = (1 - H(e^{j\mu}, e^{j\nu}))Y(e^{j\mu}, e^{j\nu})$

  Therefore, we know that $Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$
- Since X_s is white noise, we know that $S_x(e^{j\mu}, e^{j\nu}) = \sigma^2$.
- So the power spectrum of the AR process is given by

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\sigma^2}{|1 - H(e^{j\mu}, e^{j\nu})|^2}$$

- 2-D spectral estimation and modeling
  - Compute MMSE linear predictor $\hat{h}_s$ for Y_s.
  - Compute the prediction variance $\hat{\sigma}^2 = \sum_s |X_s|^2$, where $X_s = Y_s - h_s * Y_s$.
  - Estimate the power spectrum

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\hat{\sigma}^2}{\left|1 - \hat{H}(e^{j\mu}, e^{j\nu})\right|^2}$$

Generating AR Processes

- Select a causal prediction filter h_s.
- Apply IIR filter to white noise random process X_s

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Y_s is sometimes referred to as a white noise driven process.
- Do linear prediction filters $\hat{h}_s$ always form a stable IIR filter?
  - In 1-D, yes.
  - In 2-D, not always!
- Other problems:
  - Causal ordering of points may cause asymetric artifacts in results.
  - Complexity increases rapidly with IIR filter order P.