2-D Finite Impulse Response (FIR) Filters

- Difference equation

$$y(m,n) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l)x(m-k,n-l)$$

- For $N = 2$ - $\circ$ input points; $\times$ output point

$$\begin{array}{ccccc}
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array}$$

- Number of multiplies per output point $= (2N+1)^2$
- Transfer function

$$\begin{aligned}
H(z_1, z_2) &= \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) z_1^{-k} z_2^{-l} \\
H(e^{j\mu}, e^{j\nu}) &= \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) e^{-j(k\mu + l\nu)}
\end{aligned}$$

2-D FIR Filter Example 1

- Consider the impulse response $h(m,n) = h_1(m)h_2(n)$ where

$$\begin{aligned} 4h_1(n) &= (\cdots, 1, 2, 1, \cdots) \\ h_1(n) &= (\delta(n+1) + 2\delta(n) + \delta(n-1))/4 \end{aligned}$$

  Then $h(m,n)$ is a separable function with

$$16h(m,n) = \begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & 0 & 0 & 0 & 0 & 0 & \\ \cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\ \cdots & 0 & 2 & 4 & 2 & 0 & \cdots \\ \cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\ & 0 & 0 & 0 & 0 & 0 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

- The DTFT of $h_1(n)$ is

$$\begin{aligned} H_1(e^{j\omega}) &= e^{j\omega} + 2 + e^{-j\omega} \\ &= \frac{1}{2}(1 + \cos(\omega)) \end{aligned}$$

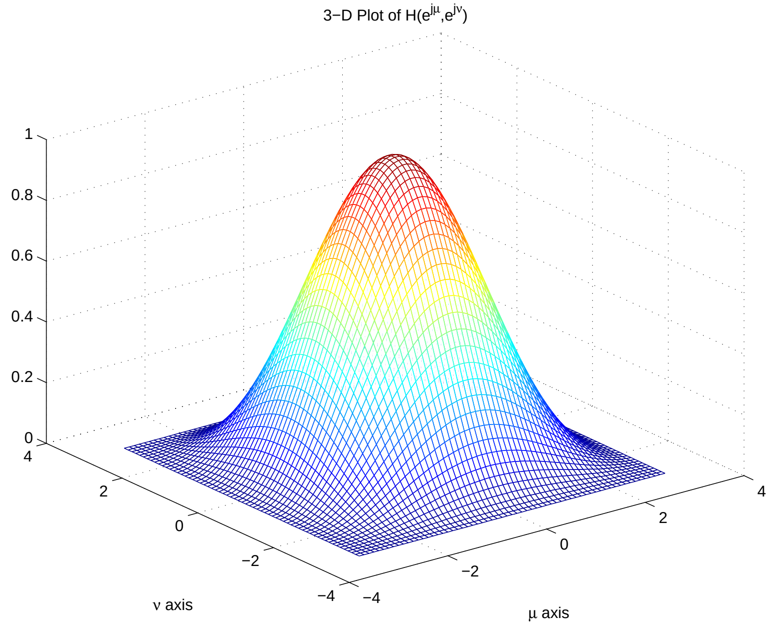
- The DSFT of $h(m,n)$ is

$$\begin{aligned} H(e^{j\mu}, e^{j\nu}) &= H_1(e^{j\mu})H_1(e^{j\nu}) \\ &= \frac{1}{4}(1 + \cos(\mu))(1 + \cos(\nu)) \end{aligned}$$

Frequency Response of 2-D FIR Filter Example 1

- Plot of frequency response

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4}\left(1 + \cos(\mu)\right)\left(1 + \cos(\nu)\right)$$



- This is a low pass filter with $H(e^{j0}, e^{j0}) = 1$

2-D FIR Filter Example 2

- Consider the impulse response $h(m,n) = h_1(m)h_2(n)$ where

$$\begin{aligned} 4h_1(n) &= (\cdots, 1, 2, 1, \cdots) \\ h_1(n) &= (\delta(n+1) - 2\delta(n) + \delta(n-1))/4 \end{aligned}$$

Then $h(m,n)$ is a separable function with

$$16h(m,n) = \begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & 0 & 0 & 0 & 0 & 0 & \\ \cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & 0 & -2 & 4 & -2 & 0 & \cdots \\ \cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\ & 0 & 0 & 0 & 0 & 0 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

- The DTFT of $h_1(n)$ is

$$\begin{aligned} H_1(e^{j\omega}) &= e^{j\omega} + 2 + e^{-j\omega} \\ &= \frac{1}{2}(1 - \cos(\omega)) \end{aligned}$$

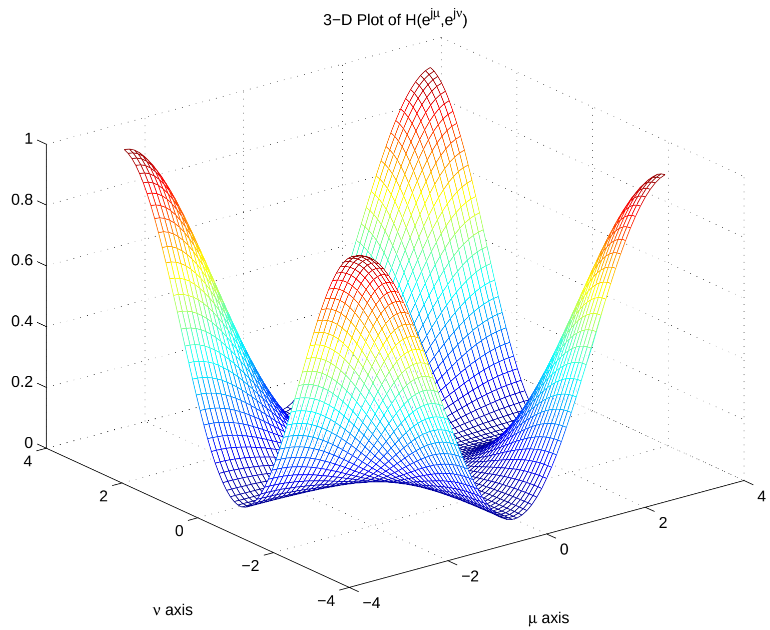
- The DSFT of $h(m,n)$ is

$$\begin{aligned} H(e^{j\mu}, e^{j\nu}) &= H_1(e^{j\mu})H_1(e^{j\nu}) \\ &= \frac{1}{4}(1 - \cos(\mu))(1 - \cos(\nu)) \end{aligned}$$

Frequency Response of 2-D FIR Filter Example 2

- Plot of frequency response

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4}\left(1 - \cos(\mu)\right)\left(1 - \cos(\nu)\right)$$



- This is a **high pass** filter with $H(e^{j0}, e^{j0}) = 0$

Ordering of Points in a Plane

- Recursive filter implementations require the ordering of points in the plane $s = (s_1, s_2)$ and $r = (r_1, r_2)$.
- Quarter plane
  - For $s, r \in \mathcal{Z}^2$, then $s < r$ means $(s_2 \leq r_2)$ and $(s_1 \leq r_1)$ and $s \neq r$.

$$\begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \times \end{array}$$

- Symmetric half plane
  - For $s, r \in \mathcal{Z}^2$, then $s < r$ means $(s_2 < r_2)$.

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ & & \times & & \end{array}$$

- Nonsymmetric half plane
  - For $s, r \in \mathcal{Z}^2$, then $s < r$ means $(s_2 < r_2)$ or $((s_2 = r_2)$ and $(s_1 < r_1))$.

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \times & & \end{array}$$

2-D Infinite Impulse Response (IIR) Filters

- Difference equation

$$
\begin{aligned}
y(m,n) &= \sum_{k=-N}^{N} \sum_{l=-N}^{N} b(k,l)x(m-k,n-l) \\
&+ \sum_{k=-P}^{P} \sum_{l=1}^{P} a(k,l)y(m-k,n-l) + \sum_{k=1}^{P} a(k,0)y(m-k,n)
\end{aligned}
$$

  Simplified notation

$$
y_s = \sum_r b_r x_{s-r} + \sum_{r>(0,0)} a_r y_{s-r}
$$

- For nonsymetric half plane with $N=0$ and $P=2$

$$
\begin{array}{ccccc}
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & &
\end{array}
$$

- Number of multiplies per output point $= (2N+1)^2 + (P+2)P$
- Transfer function

$$
\begin{aligned}
H(z_1,z_2) &= \frac{\sum_{k=-N}^{N} \sum_{l=-N}^{N} b(k,l) z_1^{-k} z_2^{-l}}{1 - \sum_{k=-P}^{P} \sum_{l=1}^{P} a(k,l) z_1^{-k} z_2^{-l} - \sum_{k=1}^{P} a(k,0) z_1^{-k}} \\
H(e^{j\mu},e^{j\nu}) &= \frac{\sum_{k=-N}^{N} \sum_{l=-N}^{N} b(k,l) e^{-j(k\mu+l\nu)}}{1 - \sum_{k=-P}^{P} \sum_{l=1}^{P} a(k,l) e^{-j(k\mu+l\nu)} - \sum_{k=1}^{P} a(k,0) e^{-j(k\mu)}}
\end{aligned}
$$

2-D IIR Filter Example 1

- Consider the difference equation

$$y(m,n) = x(m,n) + ay(m-1,n) + ay(m,n-1)$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$\begin{array}{cc} & \circ \\ \circ & \times \end{array}$$

- The transfer functions is then

$$\begin{aligned} H(z_1, z_2) &= \frac{1}{1 - az_1^{-1} - az_2^{-1}} \\ H(e^{j\mu}, e^{j\nu}) &= \frac{1}{1 - ae^{-j\mu} - ae^{-j\nu}} \end{aligned}$$

2-D IIR Filter Example 2

- Consider the difference equation

$$y(m,n) = x(m,n) + ay(m-1,n) + ay(m,n-1) + 2ay(m+1,n-1)$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$\begin{array}{ccc} & \circ & \circ \\ \circ & \times & \end{array}$$

- The transfer functions is then

$$\begin{aligned} H(z_1,z_2) &= \frac{1}{1 - az_1^{-1} - az_2^{-1} - 2az_1^{+1}z_2^{-1}} \\ H(e^{j\mu},e^{j\nu}) &= \frac{1}{1 - ae^{-j\mu} - ae^{-j\mu} - 2ae^{+j\mu-j\nu}} \end{aligned}$$

2-D IIR Filter Example 3

- Consider the difference equation

$$y(m,n) = x(m,n) + ay(m-1,n) + ay(m,n-1) + ay(m+1,n) + ay(m,n+1)$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \\ & \circ & \end{array}$$

- Theoretically, the transfer functions is then

$$\begin{aligned} H(z_1, z_1) &= \frac{1}{1 - az_1^{-1} - az_2^{-1} - az_1 - az_2} \\ H(e^{j\mu}, e^{j\nu}) &= \frac{1}{1 - ae^{-j\mu} - ae^{-j\mu} - ae^{j\mu} - ae^{j\nu}} \end{aligned}$$

- **THIS DOESN'T WORK**