Continuous Time Fourier Transform (CTFT)

$$
\begin{aligned}
F(f) &= \int_{-\infty}^{\infty} f(x) e^{-j2\pi f x} dx \\
f(x) &= \int_{-\infty}^{\infty} F(f) e^{j2\pi f x} df
\end{aligned}
$$

Useful signal definitions:

$$
\begin{aligned}
\mathrm{rect}(x) &= \begin{cases} 1 & |x| \le 1/2 \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{sgn}(x) &= \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \\
\mathrm{sinc}(x) &= \frac{\sin(\pi x)}{\pi x} \\
\Lambda(x) &= \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}
\end{aligned}
$$

The "function" $\delta(t)$ is actually an operator defined such that for all continuous and integrable functions $g(t)$ the following holds.

$$
g(0) = \int_{-\infty}^{\infty} \delta(t) g(t) dt
$$

Useful CTFT Relations

$$
\begin{aligned}
\mathrm{rect}(t) &\overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f) \\
\mathrm{sinc}(t) &\overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f) \\
\Lambda(t) &\overset{CTFT}{\Leftrightarrow} \mathrm{sinc}^2(f) \\
\delta(t) &\overset{CTFT}{\Leftrightarrow} 1 \\
1 &\overset{CTFT}{\Leftrightarrow} \mathrm{delta}(t)
\end{aligned}
$$

Continuous Space Fourier Transform (CSFT)

$$
\begin{aligned}
F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \\
f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv
\end{aligned}
$$

Comments:

1) Usually, x is horizontal and y is vertical coordinate
2) Usually, y points down.
3) Television and english text goes left to right
and top to bottom. This is reference to as *raster* order.
4) u corresponds to horizontal frequency components.
4) v corresponds to vertical frequency components.

- Useful signal definitions:

$$
\begin{aligned}
\mathrm{rect}(x,y) &\triangleq \mathrm{rect}(x)\mathrm{rect}(y) \\
\mathrm{sinc}(x,y) &\triangleq \mathrm{sinc}(x)\mathrm{sinc}(y) \\
\delta(x,y) &\triangleq \delta(x)\delta(y) \\
\mathrm{circ}(x,y) &\triangleq \mathrm{rect}(\sqrt{x^2+y^2})
\end{aligned}
$$

CSFT Properties Inherited from CTFT

Property	Functions $f(x,y)$ and $g(x,y)$	CSFT $F(u,v)$
Linearity	$af(x,y)+bg(x,y)$	$aF(u,v)+bG(u,v)$
Conjugation	$f^*(x,y)$	$F^*(-u,-v)$
Scaling	$f(ax,by)$	$\frac{1}{\lvert ab\rvert}F(u/a,v/b)$
Shifting	$f(x-x_0,y-y_0)$	$\exp\left\{-j2\pi(ux_0+vy_0)\right\}F(u,v)$
Modulation	$\exp\left\{-j2\pi(u_0x+v_0y)\right\}f(x,y)$	$F(u-u_0,v-v_0)$
Convolution	f(x,y) * g(x,y)	F(u,v) G(u,v)
Multiplication	f(x,y) g(x,y)	F(u,v) * G(u,v)
Inner Product	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)g^*(x,y)dxdy$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)G^*(u,v)dudv$

Properties Specific to CSFT

Property	Functions $f(x,y)$ and $g(x,y)$	CSFT $F(u,v)$
Separability	$f(x)g(y)$	$F(u)G(v)$
Rotation	$f\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$	$\lvert\mathbf{A}\rvert^{-1}\tilde{F}\left([u,v]\mathbf{A}^{-1}\right)$

Separability of CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx\right]e^{-j2\pi vy}dy \end{aligned}$$

Define the CTFT of $f(x,y)$ in the variable x

$$\tilde{F}(u,y) = \int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx$$

Then the CSFT may be computed as the CTFT of $\tilde{F}(u,y)$ in y

$$F(u,v) = \int_{-\infty}^{\infty} \tilde{F}(u,y)e^{-j2\pi vy}dy$$

- Comment: 2-D CSFT can be computed as two 1-D CTFT's.

CSFT of Separable Functions

Let

$$\begin{aligned} g(t) &\overset{CTFT}{\Leftrightarrow} G(f) \\ h(t) &\overset{CTFT}{\Leftrightarrow} H(f) \end{aligned}$$

Then

$$g(x)h(y) \overset{CSFT}{\Leftrightarrow} G(f)H(f)$$

Example, let $f(x,y) = \text{rect}(x,y)$, then

$$\begin{aligned} F(u,v) &= \mathcal{F}\left\{\text{rect}(x,y)\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}(x,y)e^{-j2\pi(ux+vy)}dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}(x)\text{rect}(x)e^{-j2\pi ux}e^{-j2\pi vy}dxdy \\ &= \left[\int_{-\infty}^{\infty} \text{rect}(x)e^{-j2\pi ux}dx\right]\left[\int_{-\infty}^{\infty} \text{rect}(y)e^{-j2\pi vy}dy\right] \\ &= \text{sinc}(u)\text{sinc}(v) \\ &= \text{sinc}(u,v) \end{aligned}$$