EE 637 Exam #1

November 24, Fall 1998

Name: ______________________________

Instructions: One hour exam, with no books, notes or calculators.

Problem 1.(25pt)

The following problem is on linear minimum mean squared error (MMSE) estimation.
Let X_n and Y_n be two 1-D stationary second order random processes. Define

$$Z_n = [X_{n-p}, \cdots, X_n, \cdots, X_{n+p}]$$

and let θ be a $(2p+1) \times 1$ vector. Consider all linear estimates of Y_n with the form

$$\hat{Y}_n = Z_n \theta \ . \tag{1}$$

a) Compute an expression for the MMSE linear estimate of Y_n.

b) Consider the case were $X_n = Y_n * h_n$ where h_n is a linear filter and Y_n is a white noise sequence with $E[Y_n] = 0$ and $E[(Y_n)^2] = \sigma^2$. Calculate an expression for the optimal θ in terms of h_n and σ^2.

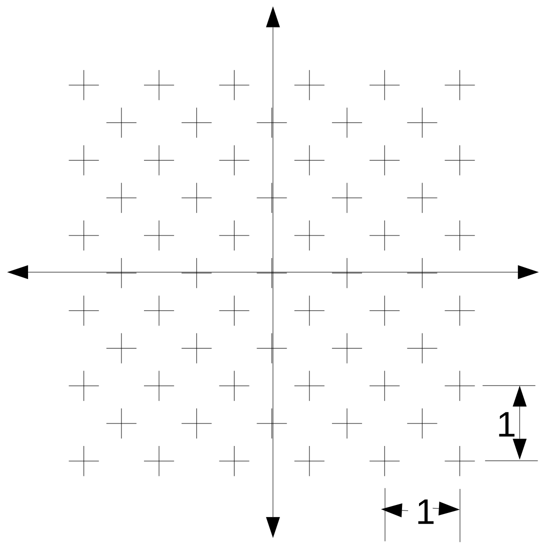
c) Define the sequence θ_n so that

$$\hat{Y}_n = \sum_{i=-p}^{p} X_{n-i} \theta_i \ .$$

Then the values of the sequence θ_n define the MMSE linear predictor. Using the assumptions of part b), calculate an expression for the DTFT of θ_n as $p \to \infty$.

Problem 2.(25pt)

In this problem, you will anaylze the effect of a non-rectangular sampling grid. Consider a 2-D function $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ where $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ is a continuously valued vector. For a particular application, $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is sampled on the non-rectangular grid shown below.



Also define the function

$$\tilde{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(A\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

where A is a 2×2 matrix.

a) Find an orthonormal matrix A so that the sampling of $\tilde{f}$ is a rectangular grid.

b) Give sufficient conditions on $\tilde{F}(u, v)$ the 2-D Fourier transform of $\tilde{f}(x, y)$, that insure that the signal $\tilde{f}(x, y)$ can be reconstructed. Sketch your conditions as a figure. (Please choose the simplest possible conditions to make things easy.)

c) Give sufficient conditions on $F(u, v)$ the 2-D Fourier transform of $f(x, y)$, that insure that the signal $f(x, y)$ can be reconstructed. Sketch your conditions as a figure. (Please choose the simplest possible conditions to make things easy.)

d) Why would this non-rectangular sampling grid be useful?

Problem 3.(25pt)

Consider the X, Y, Z to L, a, b transformation given by

$$\begin{aligned}\begin{bmatrix} L \\ a \\ b \end{bmatrix} &= F(X,Y,Z) \\ &= \begin{bmatrix} 100Y^{1/3} \\ 500(X^{1/3} - Y^{1/3}) \\ 200(Y^{1/3} - Z^{1/3}) \end{bmatrix}\end{aligned}$$

In an application, a color X, Y, Z is reproduced with some error as $X + \Delta X$, $Y + \Delta Y$, $Z + \Delta Z$.

a) Find an expression for a matrix B so that

$$\begin{aligned}\Delta E^2 &= (\Delta L)^2 + (\Delta a)^2 + (\Delta b)^2 \\ &= ||F(X + \Delta X, Y + \Delta Y, Z + \Delta Z) - F(X,Y,Z)||^2 \\ &\approx ||\,[\Delta X, \Delta Y, \Delta Z]\,||_B^2\end{aligned}$$

b) Let X be a achromatic image in the range 0-255 which is gamma corrected with $\gamma = 1.8$. Let Y be the same image, but stored with $\gamma = 2.2$. Calculate an expression for transformation from X to Y.

Problem 4.(25pt)

a) A lossy achromatic image coder works by taking blocks of N pixels and applying vector quantization (VQ). The indices of the VQ codebook are then binary encoded and stored. Let b be the number of bits per pixel required to store the encoded image, and let M be the number of codewords in the VQ codebook. Calculate an expression for the size of the codebook as a function of the block size N and the bit rate b.

b) Let the values $q_j \in \mathbb{R}$ be quantization levels (codewords) for a 1-D quantizer that attempts to minimize the total distortion

$$Error = \sum_{i=1}^{N} \min_{j=1,\cdots,M} |x_i - q_j| \ .$$

Derive the explicit expressions for the steps of a LBG algorithm to minimize this error. List the precise steps as an algorithm.