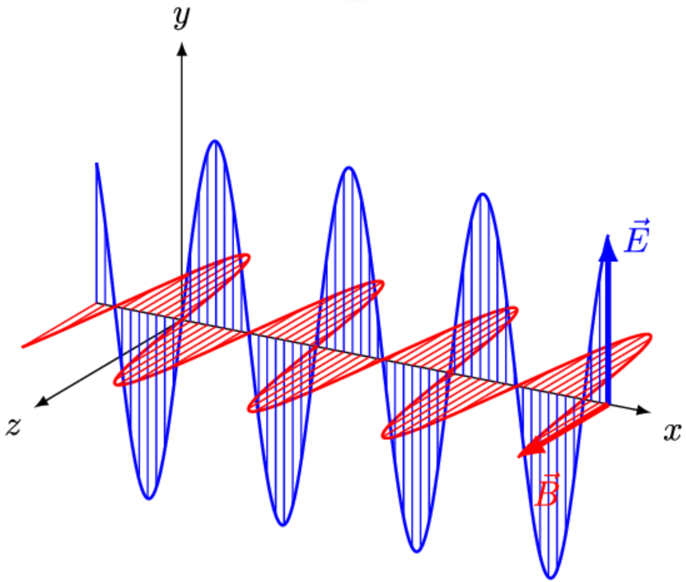
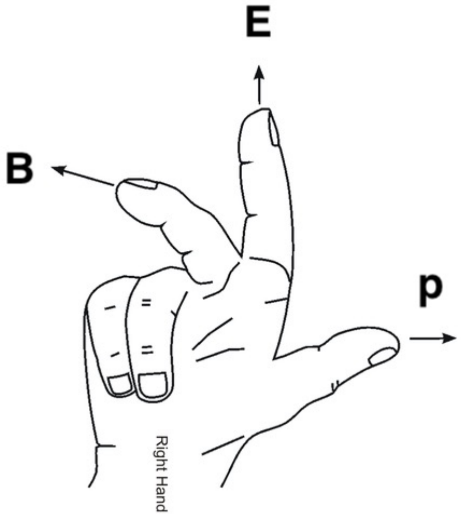
Notes on Fourier Optics

- How does light propagate through free space?
- What do lenses do?
- How does a directional antenna work?
- Four important ideas:
  - Wave equation
  - Optical plane wave and direction of transmission/arrival
  - Fresnel approximation: Assumes light is traveling through an aperture
  - Fraunhofer approximation: Assumes light is far from the aperture

Electro-Magnetic Waves



- EM waves are vector fields
  - E and B are traveling vector fields
  - E is perpendicular to B
  - $\vec{p} = E \times B$ where $\vec{p}$ is the direction of travel



- Plane linearly polarized wave

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The Wave Equation

- From Maxwell's equations:
  - We can show that

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

This is the wave equation in 3 dimensions.

  - For simplicity, we consider a single component of the electric field vector:

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

  - where μ is the permittivity; and ϵ is the dielectric constant; and $c = \frac{1}{\sqrt{\mu\varepsilon}}$ where c is the speed of light in the medium.

Solution to the Wave Equation in Time

- The wave equation is given by

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

- Separation of variables gives solutions of the form

$$E_t(x,y,z,t) = E_o \exp\left[j2\pi\left(v_x x + v_y y + v_z z\right)\right]\exp\left[-j2\pi f_o t\right]$$
$$= E_o \exp\left[j2\pi\left(v_x x + v_y y + v_z z - f_o t\right)\right]$$

  – where

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = v_o = \frac{1}{\lambda} = \frac{f_o}{c}$$

$$\lambda = \frac{1}{v_o} = \frac{c}{f_o}$$

Wavelength (m) → λ

spatial frequency (m^{-1}) → v_o

Velocity (m/sec) → c

Frequency (Hz) → f_o

- We will assume that f_o, v_o, and c are all fixed.

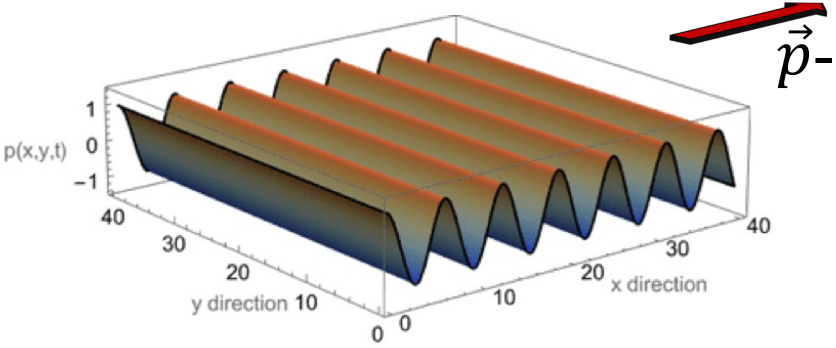
Plane Waves

- Plane waves have the form

$$E_t(x, y, z, t) = E_o \exp\left[j2\pi\left(v_x x + v_y y + v_z z\ \right)\right] \exp[-j2\pi f_o t]$$

$$= E_o \exp[j2\pi(v_o(\vec{p} \cdot \vec{r}) - f_o t\ )]$$

  – where $\vec{r} = [x, y, z]$, $\vec{v} = \left[v_x, v_y, v_z\right]$, $v_o = \|\vec{v}\|$ and $\vec{p} = \frac{\vec{v}}{\|\vec{v}\|}$.



$\vec{p}$-direction of travel

$$v_o = \frac{f_o}{c}$$

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Plane Wave Interpretation

- Solution has the form:

$$E_t(x,y,z,t) = E_o \exp\left[j2\pi\left(v_x x + v_y y + v_z z\right)\right] \underbrace{\exp[-j2\pi f_o t]}_{\text{carrier wave}}$$

- Electric field is the real component:

$$\text{Electric Field} = \text{Re}\{E_t(x,y,z,t)\}$$

- So, we can always recover the space-time solution from:

$$E(x,y,z) = E_o \exp\left[j2\pi\left(v_x x + v_y y + v_z z\right)\right]$$

  - where

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = v_o = \frac{1}{\lambda}$$

1D Plane Wave Interpretation

- Solution has the form:

$$E_t(x,t) = E_o \exp[j2\pi(v_o x)]\exp[-j2\pi f_o t]$$

$$= E_o \exp\left[j2\pi v_o\left(x - \frac{v_o}{f_o}t\right)\right]$$

Velocity (m/sec)

- Then the electric field is given by

$$\text{Electric Field} = \text{Re}\{E_t(x,y,z,t)\}$$

$$= \text{Re}\left\{E_o \exp\left[j2\pi v_o\left(x - \frac{v_o}{f_o}t\right)\right]\right\}$$

$$= A_o \text{Re}\left\{\exp\left[j2\pi v_o\left(x - \frac{v_o}{f_o}t\right) + j\theta\right]\right\}$$

$$= A_o \cos\left\{2\pi v_o\left(x - \frac{v_o}{f_o}t\right) + \theta\right\}$$

$$E_o = A_o e^{j\theta}$$

- So the plane wave is completely specified by

$$E(x,t) = E_o \exp[j2\pi(v_o x)]$$

2D Solutions

- In 3D, we have that

$$E(x, y, z) = E_o \exp\left[j2\pi\left(v_x x + v_y y + v_z z\right)\right]$$

  – where

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = v_o = \frac{1}{\lambda}$$

- So, if $v_x = 0$, we have that

$$E(y, z) = E_o \exp\left[j2\pi\left(v_y y + v_z z\right)\right]$$

  – where

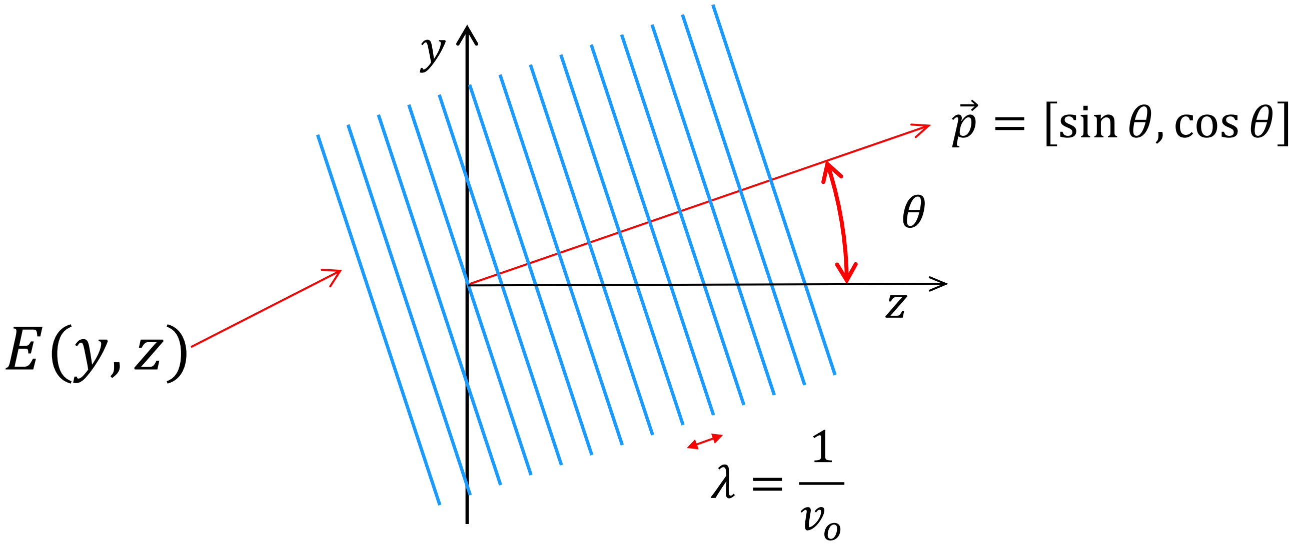
$$v_y = v_o \sin\theta$$

$$v_z = v_o \cos\theta$$

Direction of Propagation for 2D Solution

- If $v_x = 0$,

$$E(y,z) = E_o \exp[j2\pi(y\, v_o \sin\theta + z\, v_o \cos\theta)]$$



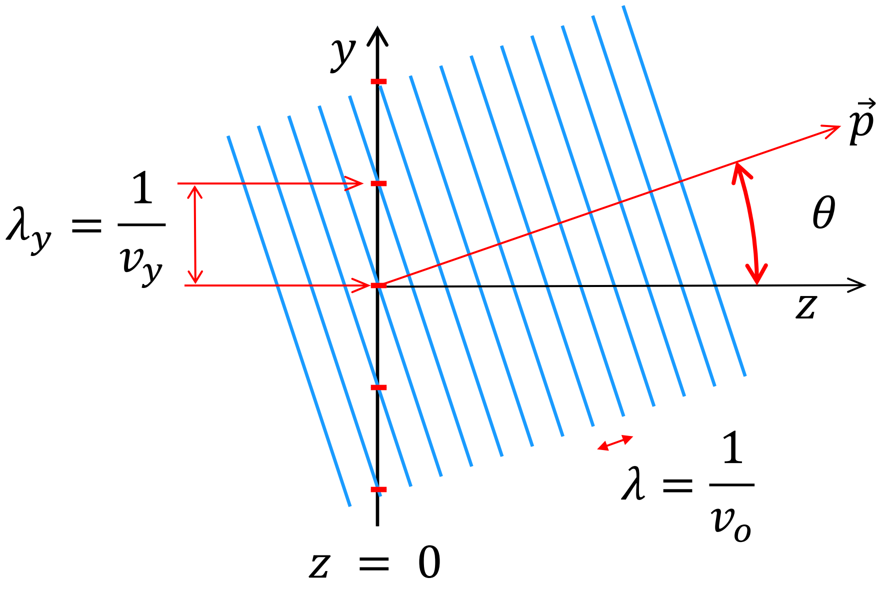
Intuition from 2D Solution

- The E field on the image plane ($z = 0$) is given by:

$$u_0(y) = E_o \exp[j2\pi(y\, v_o \sin\theta + 0\, v_o \cos\theta)]$$

$$= E_o \exp[j2\pi(y\, v_o \sin\theta)]$$

$$v_y = v_o \sin\theta$$

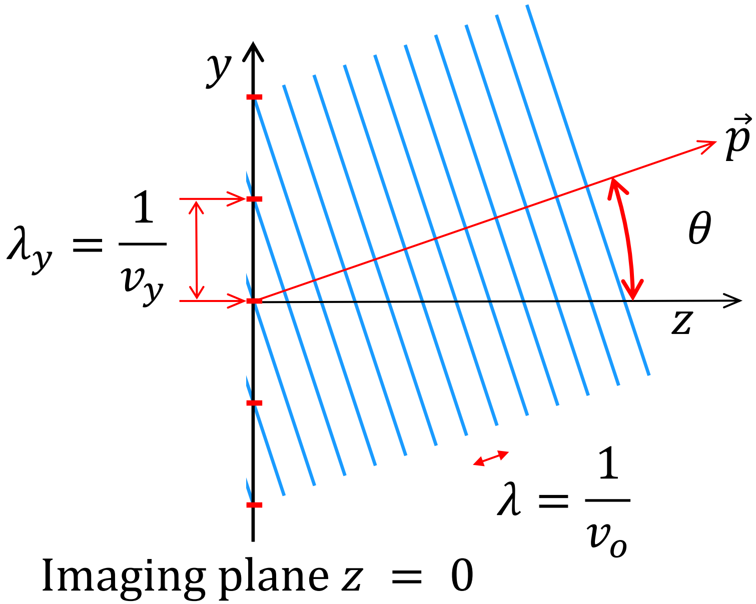


$$\lambda_y = \frac{\lambda}{\sin\theta}$$

Direction of Transmission in 2D

- A plane wave arriving at angle θ creates an electric field on the image plane given by

$$u_0(y) = E_o \exp[j2\pi(y \underbrace{v_o \sin\theta}_{v_y})]$$



- Angle of departure:

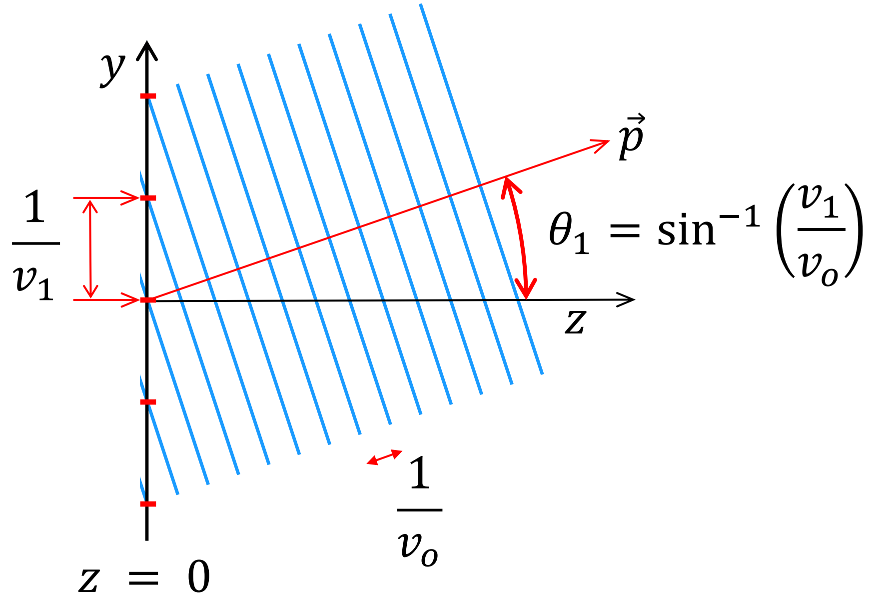
$$\theta = \sin^{-1}\left(\frac{v_y}{v_o}\right)$$

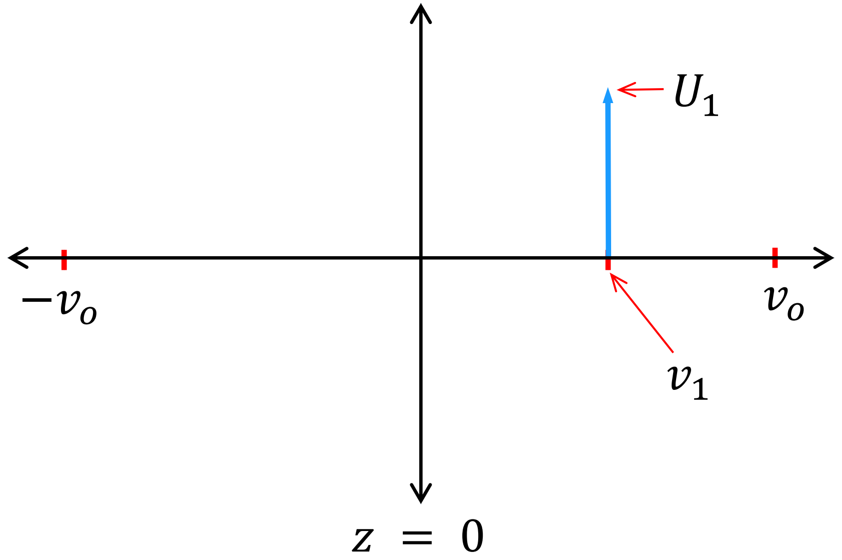
Beam Forming: One transmitted signal

- Excite a signal at $z = 0$ of the form:

$$u_0(y) = U_1 \exp[j2\pi(y\, v_1)]$$

- This "forms a beam" in the direction $\theta_1 = \sin^{-1}\left(\frac{v_1}{v_o}\right)$



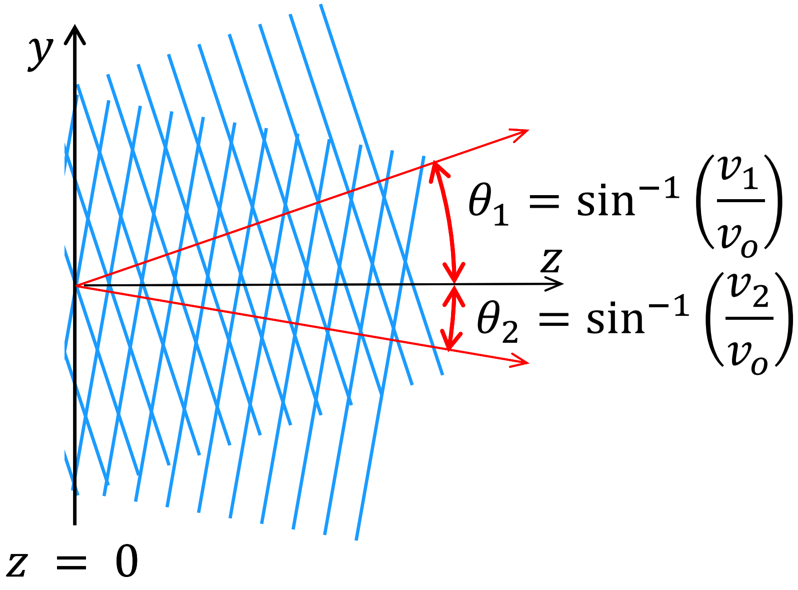


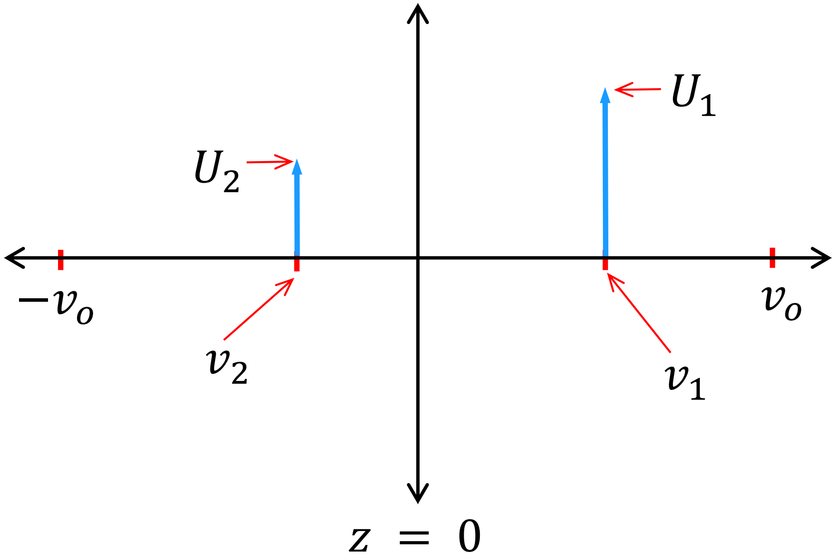
Beam Forming: Two transmitted signals

- Excite a signal at $z = 0$ of the form:

$$u_0(y) = U_1 \exp[j2\pi(y\, v_1)] + U_2 \exp[j2\pi(y\, v_2)]$$

- This "forms a beam" in the directions $\theta_1 = \sin^{-1}\left(\frac{v_1}{v_o}\right)$ and $\theta_2 = \sin^{-1}\left(\frac{v_2}{v_o}\right)$





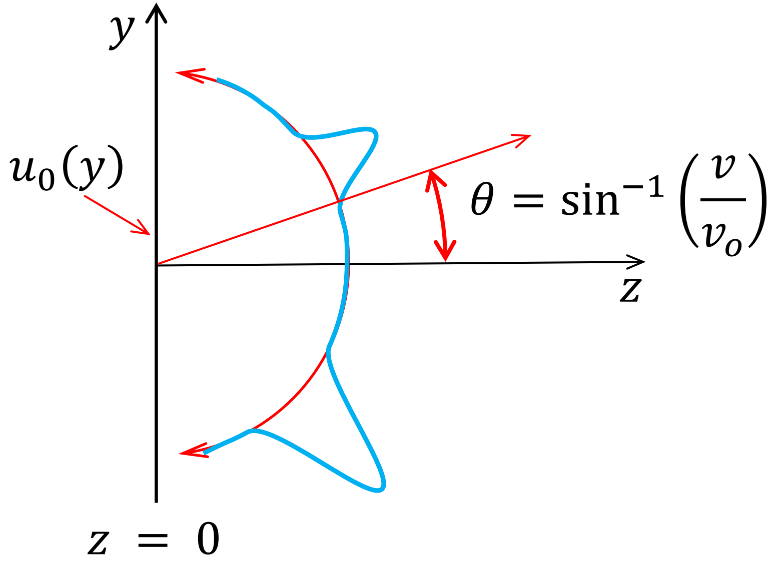
Beam Forming: Arbitrary transmitted signals

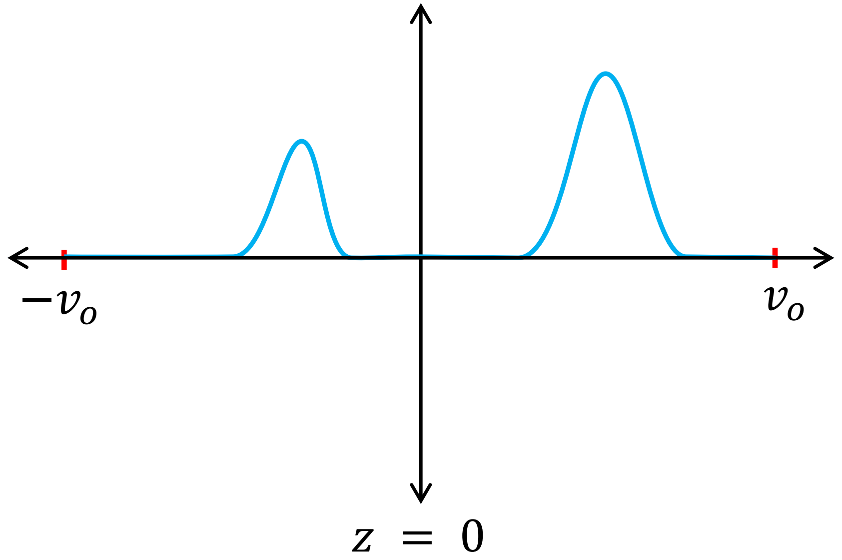
- Excite a signal at $z = 0$ of the form:

$$u_0(y) = \int_{\Re} U_0(v) \exp[j2\pi(y\,v)]\,dv$$

  – where $U_0 = \mathcal{F}[u_o]$ is the Fourier transform of u_o.

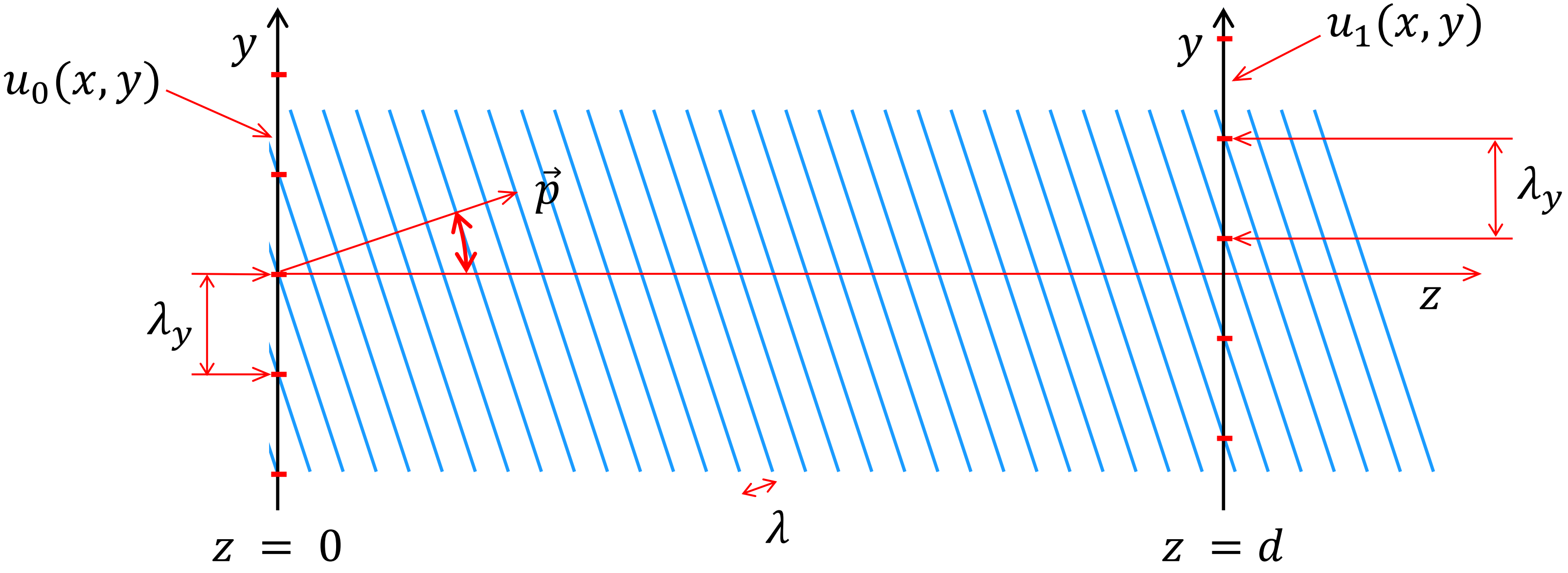
- This "forms a beam" in all directions given by





Propagation between Planes

- For a wave of the form: $E(x, y, z) = \exp[j2\pi(v_x x + v_y y + v_z z)]$
- Wave on the $z = 0$ plane is: $u_0(x, y) = \exp[j2\pi(v_x x + v_y y)]$
- Wave on the $z = d$ plane is: $u_d(x, y) = \exp[j2\pi(v_x x + v_y y + v_z d)]$

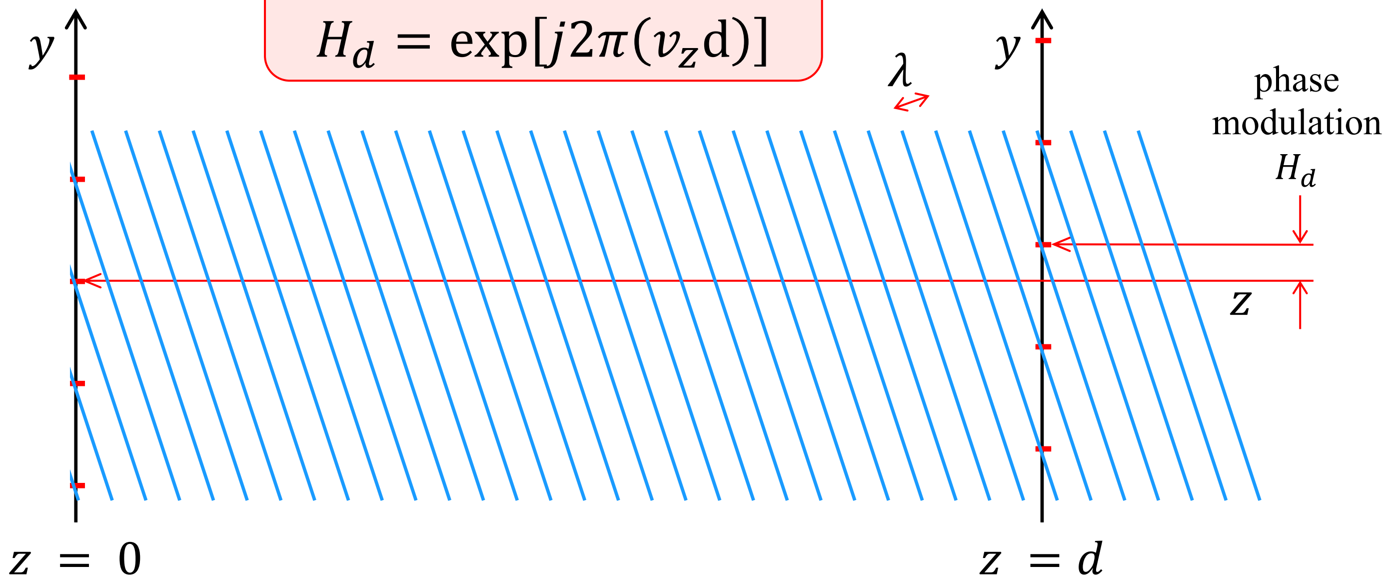


Phase Modulation between Planes

- For a wave of the form: $E(x,y,z) = \exp[j2\pi(v_x x + v_y y + v_z z)]$
- Wave on the $z = 0$ plane of: $u_0(x,y) = \exp[j2\pi(v_x x + v_y y)]$
- Wave on the $z = d$ plane of: $u_d(x,y) = \exp[j2\pi(v_x x + v_y y)]\exp[j2\pi(v_z d)]$

$$= \exp[j2\pi(v_z \mathrm{d})]\, u_0(x,y) = H_d\, u_0(x,y)$$

Phase modulation is
$H_d = \exp[j2\pi(v_z \mathrm{d})]$



Deriving the Fresnel Phase Modulation Function

- So then we have that

$$\exp[j2\pi v_z d] = \exp\left[j2\pi\left(v_o^2 - [v_x^2 + v_y^2]\right)^{\frac{1}{2}} d\right]$$

$$= \exp\left[j2\pi d v_o\left(1 - \frac{v_x^2 + v_y^2}{v_o^2}\right)^{\frac{1}{2}}\right]$$

$$\approx \exp\left[j2\pi d v_o\left(1 - \frac{v_x^2 + v_y^2}{2v_o^2}\right)\right]$$

$$= \exp[j2\pi d v_o] \exp\left[-j2\pi d v_o\left(\frac{v_x^2 + v_y^2}{2v_o^2}\right)\right]$$

$$= \exp[j2\pi d v_o] \exp\left[-j2\pi d\left(\frac{v_x^2 + v_y^2}{2v_o}\right)\right]$$

Useful Facts:

$$v_o^2 = v_x^2 + v_y^2 + v_z^2$$

For $|\beta| \ll 1$, $\sqrt{1-\beta} \approx 1 - \frac{\beta}{2}$

Assumes that:

$$\frac{v_x^2 + v_y^2}{v_o^2} \ll 1$$

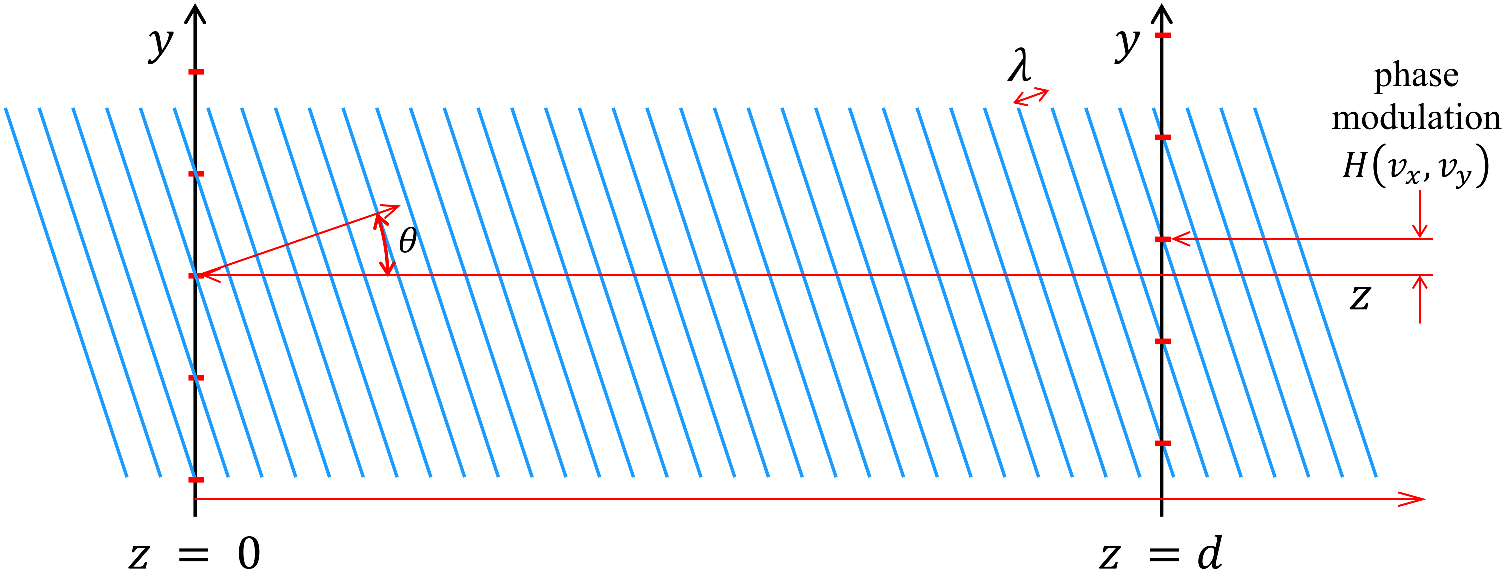
$$\sin^2(\theta) \ll 1$$

Interpretation of Fresnel Phase Modulation

- The transfer function is given by

$$H(v_x, v_y) = \exp[j2\pi d v_o] \exp\left[-j2\pi d v_o \left(\frac{v_x^2 + v_y^2}{2v_o^2}\right)\right]$$

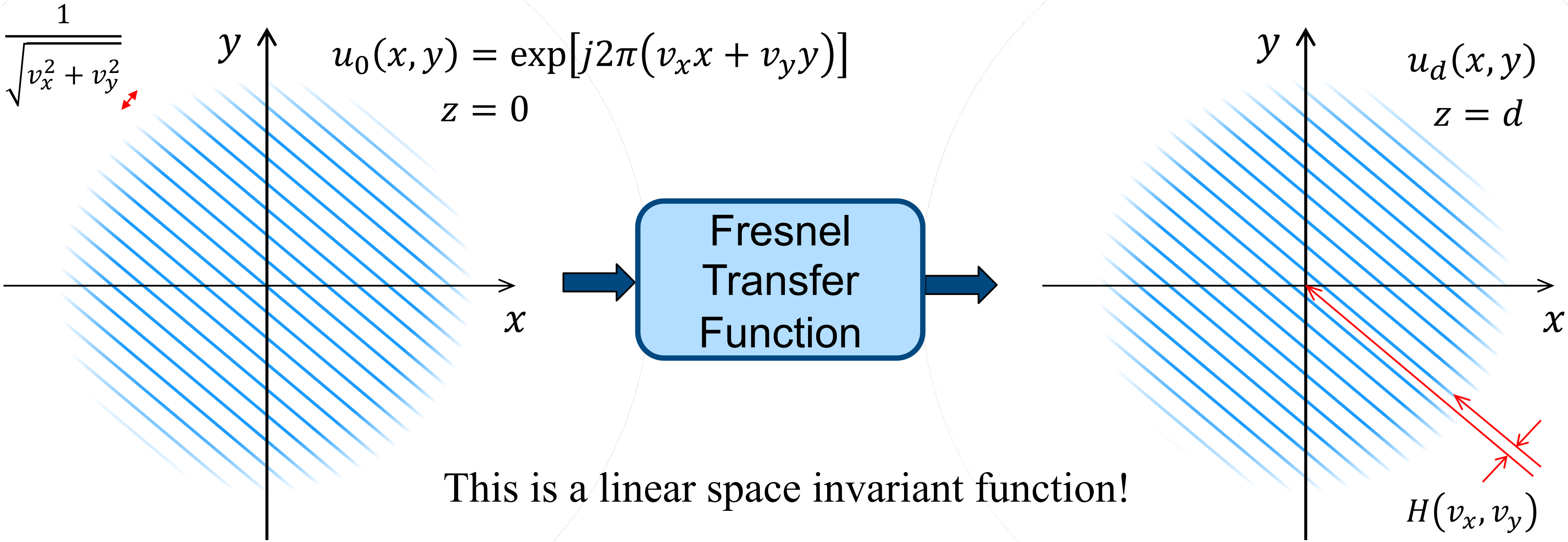
  - Pure phase shift; no amplitude; ⇒ conserves energy
  - Phase shift depends on angle $\frac{1}{2}\sin^2\theta$ where $\frac{\sqrt{v_x^2+v_y^2}}{v_o} = \sin\theta = \sin(\text{tile angle})$ in 3D.



3D Interpretation of Fresnel Phase Modulation

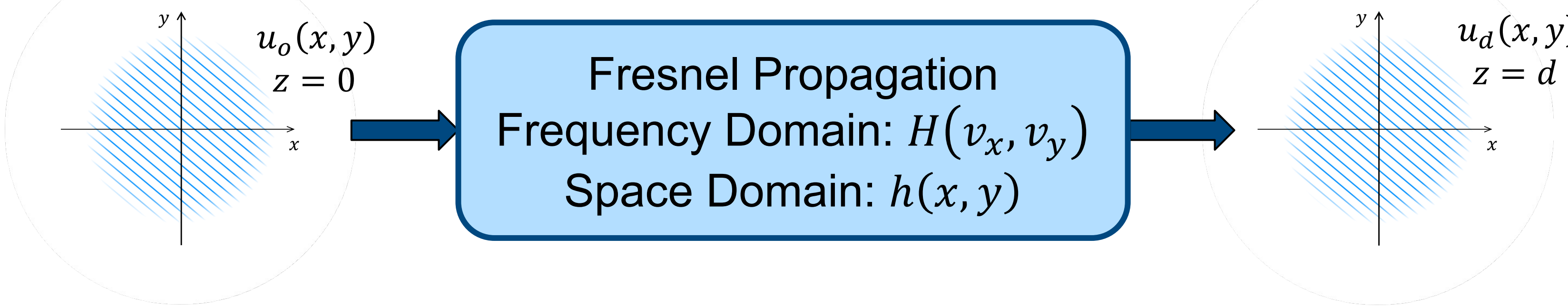
- We can visualize the plane wave on the two planes.

$$u_d(x,y) = H(v_x, v_y)\exp[j2\pi(v_x x + v_y y)]$$



This is a linear space invariant function!

Fresnel Propagation: An LSI System



- Frequency domain transfer function:

$$H(v_x, v_y) = \exp[j2\pi d v_o] \exp\left[-j2\pi d v_o \left(\frac{v_x^2 + v_y^2}{2v_o^2}\right)\right]$$

- Space domain point-spread function (psf):

$$h(x, y\,) = h_o \exp\left[j\pi\left(\frac{x^2 + y^2}{\lambda d}\right)\right]$$

(from a table of transform pairs)

$$h_o = \frac{1}{j\lambda d} \exp\left[j2\pi \frac{d}{\lambda}\right]$$

$$\lambda = \frac{1}{v_o}$$

Fresnel Transfer Function

Assumes that:

$$\frac{v_x^2 + v_y^2}{v_o^2} \ll 1$$

$$\sin^2(\theta) \ll 1$$

- In the frequency domain:

$$U_d(v_x, v_y) = H(v_x, v_y)\, U_0(v_x, v_y)$$

  – where

$$H(v_x, v_y) = \exp[j2\pi d v_o] \exp\left[-j2\pi d v_o \left(\frac{v_x^2 + v_y^2}{2v_o^2}\right)\right]$$

$$U_o(v_x, v_y) = \mathcal{F}[u_o] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x, y) \exp\{-j2\pi(v_x x + v_y y)\}\, dx\, dy$$

$$U_1(v_x, v_y) = \mathcal{F}[u_1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1(x, y) \exp\{-j2\pi(v_x x + v_y y)\}\, dx\, dy$$

Fresnel Convolution

Assumes that:

$$\frac{x^2 + y^2}{d^2} \ll 1$$

$$\sin^2(\theta) \ll 1$$

- In the space domain:

$$u_d(x, y) = h(x, y) * u_0(x, y)$$

$$u_d(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0(x', y')\, h(x - x', y - y')\, dx'\, dy'$$

  – where

$$h(x, y) = h_o \exp\left[j\pi \frac{d}{\lambda}\left(\frac{x^2 + y^2}{d^2}\right)\right]$$

$$h_o = \frac{\exp\left[j2\pi \frac{d}{\lambda}\right]}{j\lambda d}$$

Fresnel Convolution/PSF Example

- In the space domain:

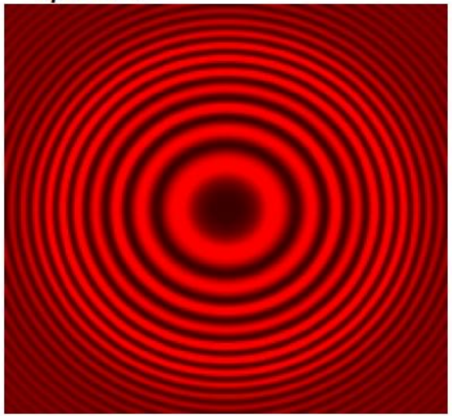
$$u_d(x, y) = h(x, y) * u_0(x, y)$$

  – where

$$h(x, y) = h_o \exp\left[j\pi \frac{d}{\lambda}\left(\frac{x^2 + y^2}{d^2}\right)\right]$$

$$h_o = \frac{\exp\left[j2\pi \frac{d}{\lambda}\right]}{j\lambda d}$$

Fresnel diffraction of circular aperture"



$$h(x, y) * \mathrm{circ}(x, y)$$

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Fresnel Transformation

Assumes that:

$$\frac{x^2 + y^2}{d^2} \ll 1$$

$$\sin^2(\theta) \ll 1$$

- Derivation

$$u_d(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x',y')\, h(x-x',y-y')dx'\,dy'$$

$$= h_o \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x',y') \exp\left\{j\pi \frac{(x-x')^2 + (y-y')^2}{\lambda d}\right\} dx'\,dy'$$

$$= h_o \exp\left\{j\pi \frac{x^2+y^2}{\lambda d}\right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x',y') \exp\left\{j\pi \frac{x'^2+y'^2}{\lambda d}\right\} \exp\left\{-j2\pi \frac{xx'+yy'}{\lambda d}\right\} dx'\,dy'$$

$$= h_o \exp\left\{j\pi \frac{x^2+y^2}{\lambda d}\right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{u}_0(x',y') \exp\left\{-j2\pi \frac{xx'+yy'}{\lambda d}\right\} dx'\,dy'$$

$$= h_o \exp\left\{j\pi \frac{x^2+y^2}{\lambda d}\right\} \tilde{U}_0\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$

Fresnel Transformation

Assumes that:

$$\frac{x^2 + y^2}{d^2} \ll 1$$

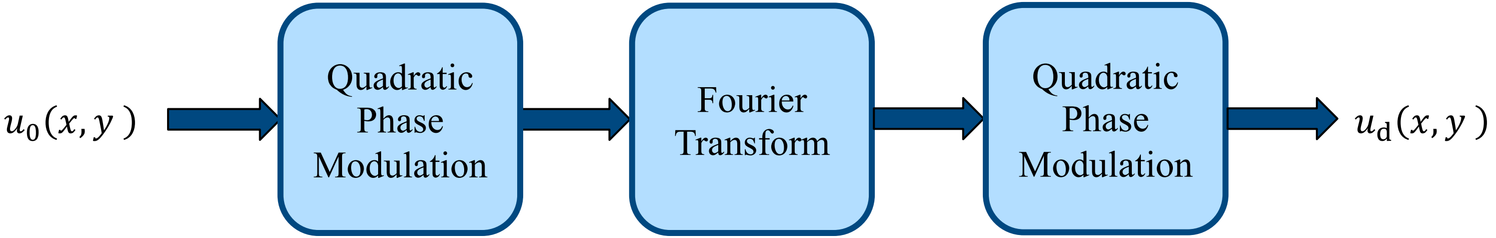
$$\sin^2(\theta) \ll 1$$

- Derivation

$$\tilde{u}_0(x, y) = \exp\left\{j\pi \frac{x^2 + y^2}{\lambda d}\right\} u_0(x, y)$$

$$\tilde{U}_0(v_x, v_y) = \mathcal{F}[\tilde{u}_o(x, y)]$$

$$u_d(x, y) = h_o \exp\left\{j\pi \frac{x^2 + y^2}{\lambda d}\right\} \tilde{U}_0\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$



Fraunhofer Propagation

Assumes that:

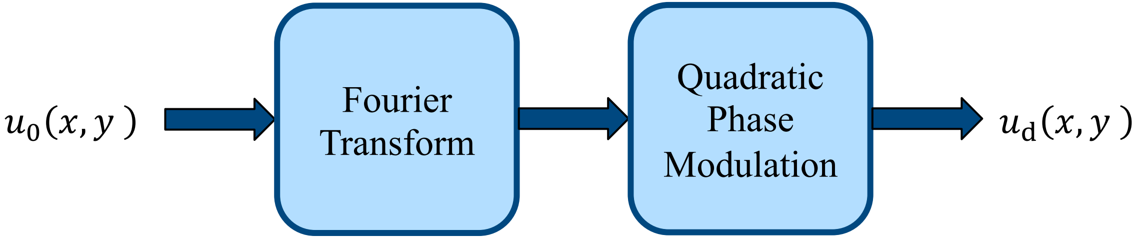
If $u_0 = 0$ for $\|[x, y]\| > a$, then $\frac{a^2}{\lambda d} \ll 1$

- Assume aperture is small, then $\tilde{u}_0(x, y) \approx u_0(x, y\,)$ and

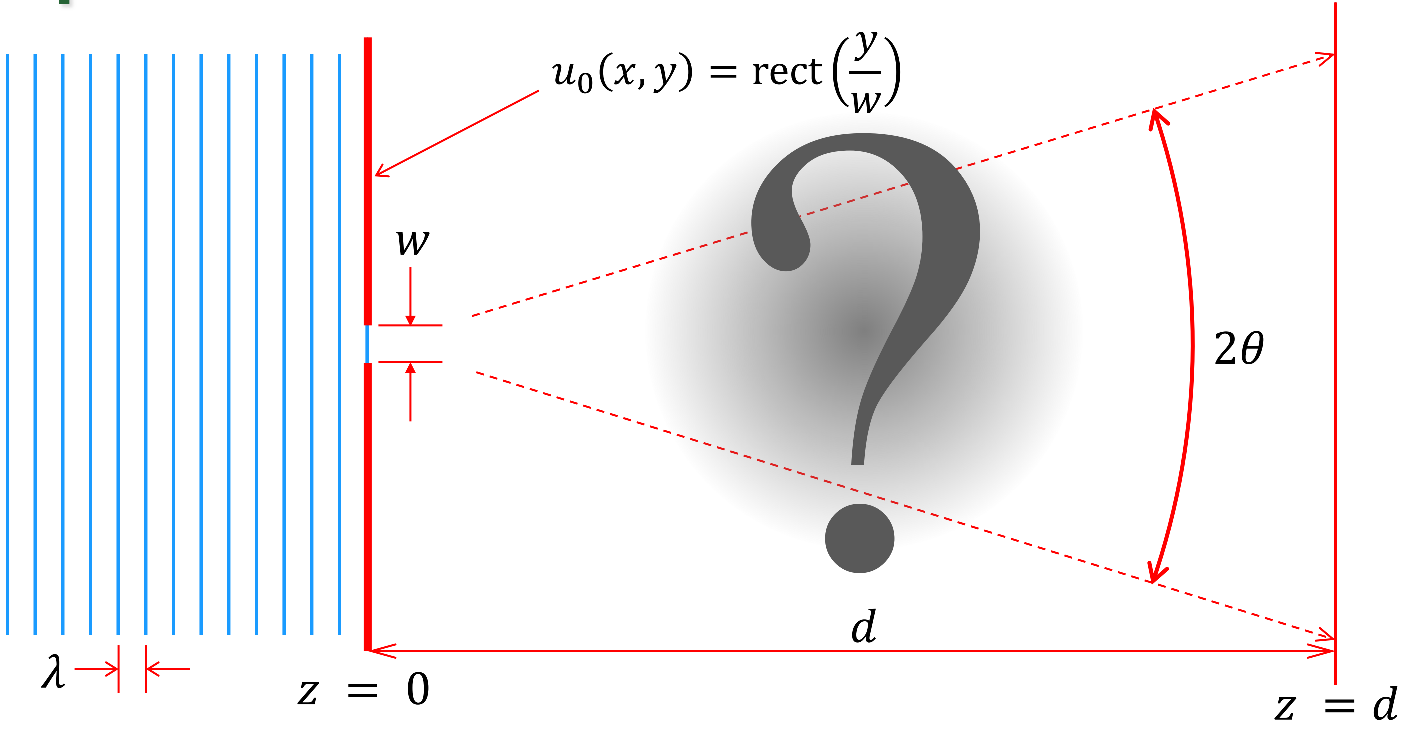
$$u_d(x, y) = h_o \exp\left\{j\pi \frac{x^2 + y^2}{\lambda d}\right\} U_0\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$

  – where

$$U_0\left(v_x, v_y\right) = \mathcal{F}[u_o(x, y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x, y) \exp\{-j2\pi\left(v_x x + v_y y\right)\}\, dx\, dy$$

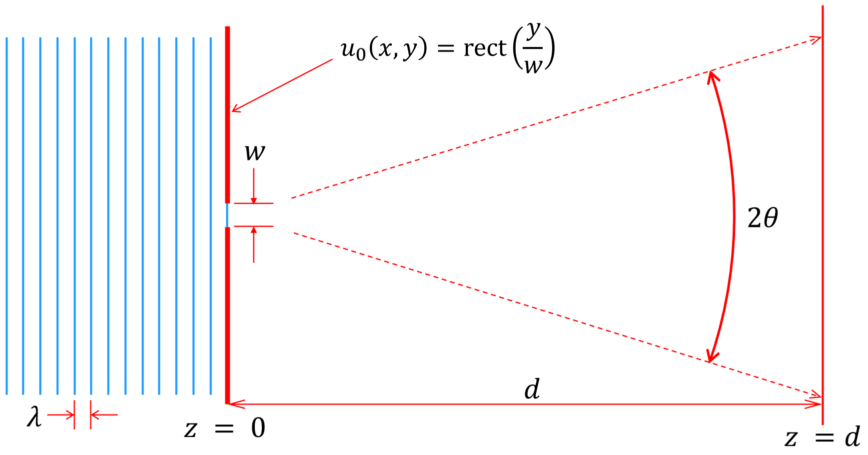


Slit Experiment



- Questions:
  - How does wave propagate?
  - What is $u_d(x,y) = ?$

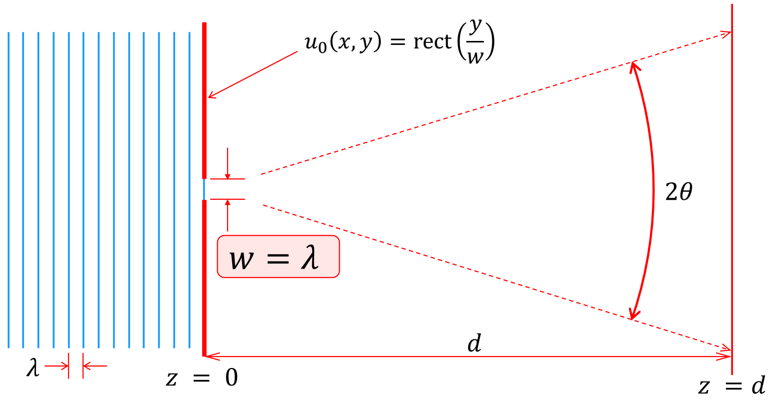
Slit Experiment: Which approximation holds?



- Fresnel:
  - Holds if $\sin^2(\theta) \ll 1$
  - If $\sin^2(\theta) = 0.05$, then $|\theta| \leq 13°$
- Fraunhofer:
  - Holds if ${w^2}/{\lambda d} \ll 1$; or equivalently, $d \gg w\left(\frac{w}{\lambda}\right)$
  - If $\lambda = 1\mu\text{m}$ and $w = 1\text{mm}$, then $d \gg 1\text{m}$
  - If $\lambda = 1\mu\text{m}$ and $w = 1\text{cm}$, then $d \gg 100\text{m}$

2D Thin Slit Analysis

- Which approximation:
  - $d \gg w\left(\frac{w}{\lambda}\right) = w \;\Rightarrow\;$ <u>Yes</u>! $\;\Rightarrow\;$ Fraunhofer



- Fraunhofer Analysis
  - Field at slit: $u_0(y) = \text{rect}\left(\frac{y}{w}\right)$
  - Field at distance d:

$$
\begin{aligned}
u_d(y) &= h_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} U_0\left(\frac{y}{\lambda d}\right) \\
&= h_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} w \,\text{sinc}\left(\frac{wy}{\lambda d}\right) \\
&= w h_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} \text{sinc}\left(\frac{y}{d}\right)
\end{aligned}
$$

  - Field at distance d:

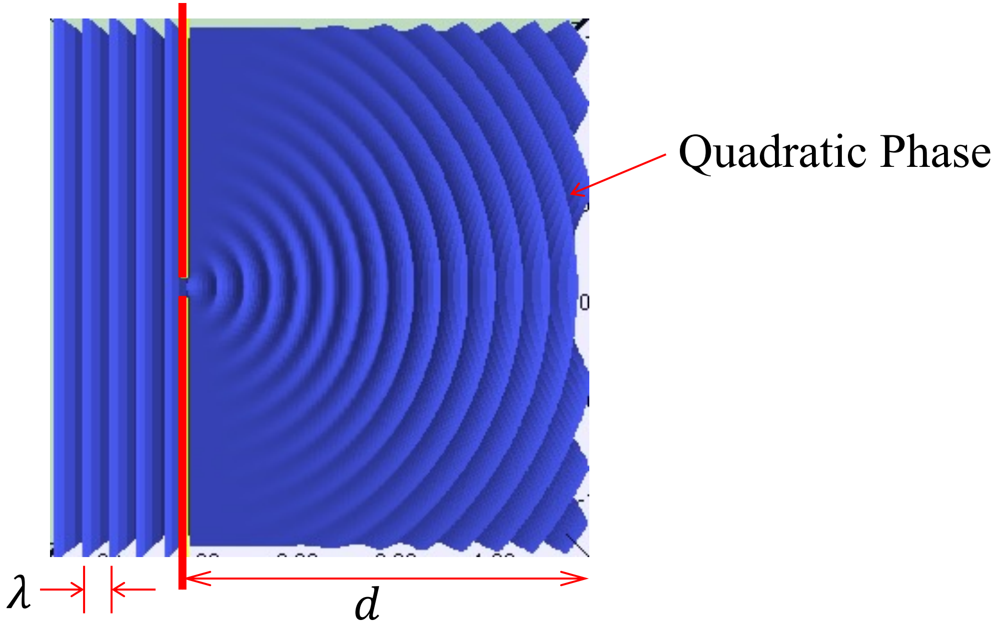
$$
\begin{aligned}
\|u_d(y)\|^2 &= \left\|U_0\left(\frac{y}{\lambda d}\right)\right\|^2 \\
&= w^2 \text{sinc}^2\left(\frac{y}{d}\right)
\end{aligned}
$$

Thin Slit Result

Example:

Thin slit

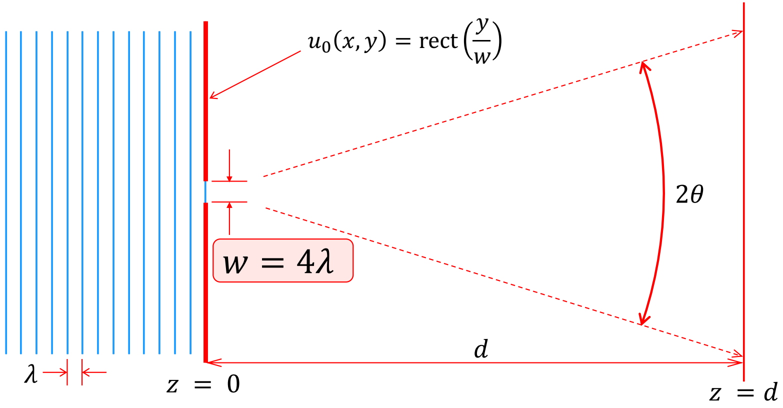
$\frac{w}{\lambda} = 1; \frac{d}{\lambda} = 15$



- Field at slit: $u_0(x, y) = \text{rect}\left(\frac{y}{w}\right)$
- Field at distance d: $u_d(x, y) = wh_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} \text{sinc}\left(\frac{y}{d}\right)$
- Power at distance d: $\|u_d(x, y)\|^2 = w^2 \text{sinc}^2\left(\frac{y}{d}\right)$

2D Wider Slit Analysis

- Which approximation:
  - $d \gg w\left(\frac{w}{\lambda}\right) = 4w \;\Rightarrow\; \underline{\text{Yes!}} \;\Rightarrow\;$ Fraunhofer



- Fraunhofer Analysis
  - Field at slit: $u_0(y) = \text{rect}\left(\frac{y}{w}\right)$
  - Field at distance d:

$$
\begin{aligned}
u_d(y) &= h_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} U_0\left(\frac{y}{\lambda d}\right) \\
&= h_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} w \,\text{sinc}\left(\frac{wy}{\lambda d}\right) \\
&= w h_o \exp\left\{j\pi \frac{y^2}{\lambda d}\right\} \text{sinc}\left(4\frac{y}{d}\right)
\end{aligned}
$$

  - Field at distance d:

$$
\begin{aligned}
\|u_d(y)\|^2 &= \left\|U_0\left(\frac{y}{\lambda d}\right)\right\|^2 \\
&= w^2 \text{sinc}^2\left(\frac{wy}{\lambda d}\right) = w^2 \text{sinc}^2\left(4\frac{y}{d}\right)
\end{aligned}
$$

First null occurs at:

$$\frac{wy}{\lambda d} = 1 \quad \Rightarrow \quad \frac{y}{d} = \frac{\lambda}{w}$$

$$y_{\text{null}} = d\left(\frac{\lambda}{w}\right)$$

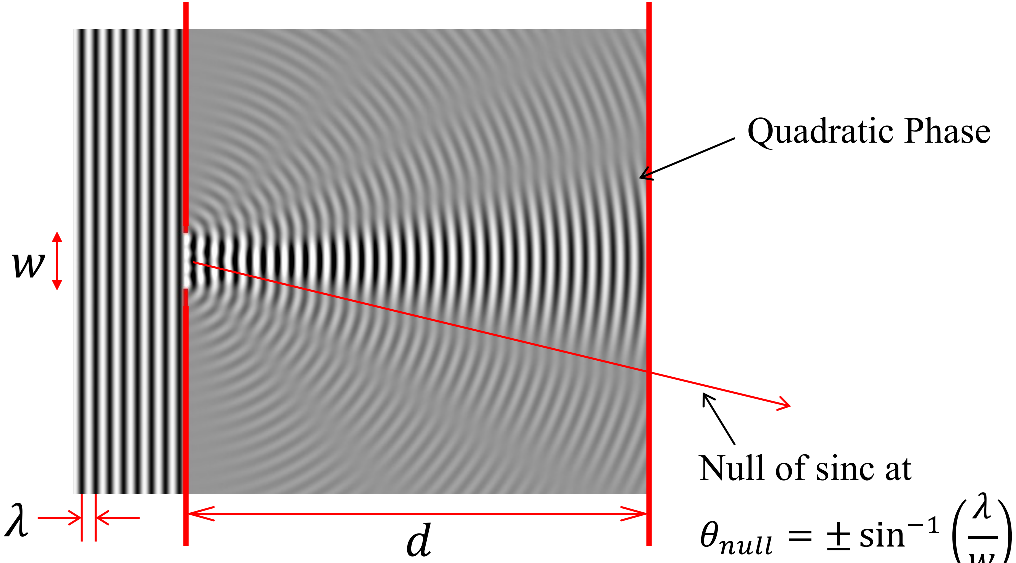
$$\theta_{\text{null}} = \sin^{-1}\left(\frac{\lambda}{w}\right)$$

Fresnel Propagation: Light through a large slit

Example:

Thin slit

$\frac{w}{\lambda} = 4; \frac{d}{\lambda} = 32$



$$\theta_{null} = \pm \sin^{-1}\left(\frac{\lambda}{w}\right) = \pm 14.47°$$

- Field at slit: $$u_0(x, y) = \text{rect}\left(\frac{y}{w}\right)$$
- Field at distance d: $$u_d(x, y) = w h_o \exp\left\{j\pi \frac{x^2}{\lambda d}\right\} \text{sinc}\left(\frac{yw}{\lambda d}\right)$$
- Power at distance d: $$\|u_d(x, y)\|^2 = w^2 \text{sinc}^2\left(\frac{yw}{\lambda d}\right)$$