Deriving the Point Spread Function (PSF) of a Thin Lens

ECE 637 Supplementary Material

Venkatesh Sridhar
Point-spread function (PSF) caused due to imperfect interference pattern

Decreasing pixel-size on detector array does not enhance resolution beyond a certain point
Lens Geometry – Parabolic Approximation

- **Parabolic approximation**: Approximate Lens surface as parabolic

  1-D case: \[ \delta_i(y) = \frac{y^2}{2R_i} \quad \text{and} \quad \delta_i(y) = \frac{y^2}{2R_i} \]

  2-D lens surface: \[ \delta_i(x,y) = \frac{x^2 + y^2}{2R_i} , \, i = 1, 2 \]

- \( R_1, R_2 \) = radius of curvature for lens surfaces
- \( n = \) refractive index of lens \((n > 1)\)
- \( L = \) Lens thickness
Phase-shift imparted by lens

• Phase change from P → P':

\[ \phi(x, y) = \frac{2\pi}{\lambda} \left( \delta_1(x, y) + \delta_2(x, y) \right) + \frac{2\pi}{(\lambda / n)} \left( L - \delta_1(x, y) - \delta_2(x, y) \right) \]

Path through Air
Path through Lens
refractive index

• Applying the parabolic approximation to \( \delta_1 \) and \( \delta_2 \)

\[ \phi(x, y) = \frac{2\pi}{\lambda} \left\{ nL - \frac{x^2 + y^2}{2} (n - 1) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right\} \]
Lens Focal length : Interpretation

• Can re-write phase-shift $\phi(x,y)$ as

$$\phi(x,y) = \frac{2\pi}{\lambda} \left\{ nL - \frac{(x^2 + y^2)}{2d_f} \right\},$$

where $d_f$ is defined as

$$\frac{1}{d_f} = (n-1) \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

• What is so special about $d_f$?

➢ Optics perspective
  – According to the famous Lensmaker’s equation, $d_f$ is the focal length of the lens

➢ Signal processing perspective
  – Typically a narrow PSF is preferable (ideal PSF is an impulse function)
  – PSF dependent on system parameters such as lens-diameter, wavelength, detector position, etc.
  – We can show that for a thin lens with sufficiently large diameter, the PSF has minimal width when detector plane is at distance $d_f$ from lens.
Interference Pattern on Detector Plane

- Huygens-Fresnel principle: each point on a wavefront is a point source

- Field $E$ at a point $P^d = (u,v,z)$ on the detector plane

$$E(u, v, z) = \frac{\alpha}{z} \int_{\text{lens}} \left( A_0 \exp\left\{ j\phi(x, y) \right\} \right) \exp\left\{ j \frac{2\pi}{\lambda} D_{P^d}^2(x, y) \right\} dx dy$$

Distance between $(x,y,0)$ and $P^d$
Binomial Approximation to Path Length

- Distance between points \((x, y, 0)\) on pupil plane and \(P^d = (u, v, z)\) on detector plane

\[
D_{pd}(x, y) = \sqrt{(x - u)^2 + (y - v)^2 + (0 - z)^2}
\]

\[
= z \sqrt{1 + \frac{(x - u)^2}{z^2} + \frac{(y - v)^2}{z^2}}
\]

- Recall binomial expansion

\[
\sqrt{1 + a} = 1 + \frac{a}{2} - \frac{a^2}{8} \cdots \approx 1 + \frac{a}{2} \quad \text{for sufficiently small } a
\]

- For sufficiently large value of \(z\)

\[
D_{pd}(x, y) \approx z \left(1 + \frac{(x - u)^2}{2z^2} + \frac{(y - v)^2}{2z^2}\right)
\]

\[
= \left[z + \frac{u^2 + v^2}{2z}\right] - \left(\frac{xu + yv}{z}\right) + \left(\frac{x^2 + y^2}{2z}\right)
\]
Calculation: Interference Pattern on Detector Plane

- Field \( E \) at a point \( P^d = (u, v, z) \) on the detector plane

\[
E(u, v, z) = \frac{A_0 \alpha}{z} \iint_{\text{Lens}} \exp \left\{ j \left( \phi(x, y) + \frac{2\pi}{\lambda} D_{P^d}(x, y) \right) \right\} dx dy
\]

\[
= \frac{A_0 \alpha}{z} \iint_{\text{Lens}} \exp \left\{ j \frac{2\pi}{\lambda} \left( nL - \frac{(x^2 + y^2)}{2d_f} \right) + \left( z + \frac{u^2 + v^2}{2z} \right) + \left( \frac{u + yv}{z} \right) + \left( \frac{x^2 + y^2}{2z} \right) \right\} dx dy
\]

- Lens mask

\[
f(x, y) = \begin{cases} 
1 & \text{if } (x, y) \text{ within lens region} \\
0 & \text{e.w.} 
\end{cases}
\]

- Re-arrange terms in above equation and change integral limits to full real space

\[
E(u, v, z) = \frac{A_0 \alpha}{z} \exp \left\{ j \frac{2\pi}{\lambda} \left( nL + z + \frac{u^2 + v^2}{2z} \right) \right\} \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} f(x, y) \exp \left\{ j \frac{2\pi}{\lambda} \left( \frac{xu + yv}{z} \right) + \left( x^2 + y^2 \right) \left( \frac{1}{z} - \frac{1}{d_f} \right) \right\} dx dy
\]

Amplitude

vanishes if \( z = d_f \)
Special Case: Interference Pattern on Detector Plane

• Let $F$ denote the continuous-space Fourier transform (CSFT) of $f$, the lens mask

$$ f(x, y) \rightarrow F(\mu, \nu) $$

• Consider special case of detector-plane position: $z = d_f$

$$ E(u, v, d_f) = \frac{A_0 \alpha}{d_f} \exp \left\{ j \frac{2\pi}{\lambda} \left( nL + z + \frac{u^2 + v^2}{2d_f} \right) \right\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp \left\{ j2\pi \left( \frac{u}{\lambda d_f} x + \frac{v}{\lambda d_f} y \right) \right\} dx dy $$

This is a 2-D Fourier Transform of $f$!!

$$ E(u, v, d_f) = \frac{A_0 \alpha}{d_f} \exp \left\{ j \frac{2\pi}{\lambda} \left( nL + z + \frac{u^2 + v^2}{2d_f} \right) \right\} F \left( \frac{-u}{\lambda d_f}, \frac{-v}{\lambda d_f} \right) $$

Can drop the – sign in front of $u$ and $v$ since $f$ is real-valued
Point Spread Function (PSF) Model

• To compute PSF we need only field intensity

\[ |E(u, v, d_f)|^2 = \left( \frac{A_0 \alpha}{d_f} \right)^2 |F \left( \frac{u}{\lambda d_f}, \frac{v}{\lambda d_f} \right)|^2 \]

• The PSF \( h(u, v) \) is then given by

\[ h(u, v) = \frac{|E(u, v, d_f)|^2}{|E(0, 0, d_f)|^2} = \frac{|F \left( \frac{u}{\lambda d_f}, \frac{v}{\lambda d_f} \right)|^2}{|F(0, 0)|^2} \]

• Note that \( F \) is dependent on shape of lens, and so, PSF \( h \) is dependent on the same
PSF of a Circular lens

- Refer to ECE 637 lecture on CSFT (2nd lecture of this course)

- For a circular lens of diameter $W$
  
  $$f(x, y) = \text{circ} \left( \frac{x}{W}, \frac{y}{W} \right)^{CSFT} \rightarrow F(\mu, \nu) = \text{jinc} (W \mu, W \nu) = \frac{J_1 \left( \pi W \sqrt{\mu^2 + \nu^2} \right)}{2W \sqrt{\mu^2 + \nu^2}}$$

  where $J_1$ is Bessel function of first kind order 1

- PSF for circular lens
  
  $$h(u, v) = \text{jinc} \left( \frac{W}{\lambda d_f} u, \frac{W}{\lambda d_f} v \right)$$

- **Intuition**: Large $(W/\lambda)$ ratio corresponds to narrow PSF !!

- **Exercise**: Why do parabolic RADAR antennas have large diameters?
References

