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AN INVERSION FORMULA FOR CONE-BEAM RECONSTRUCTION*

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Abstract. An analytic inversion formula allowing the reconstruction of a three-dimensional object from x-ray cone-beams is given. The formula is valid for the case where the source of the beams describes a bounded curve satisfying a set of weak conditions.

1. Introduction and notation. Reconstructing the density function of a three-dimensional object via x-ray projection data can be formulated mathematically as recovering the density function from its line integrals [7]. The reduction of the dimension of the problem from three to two can be done by considering three-dimensional objects as a stack of their cross-sections. This reduction allows the use of Radon's inversion formulas [15] which express a function defined on a plane in terms of its line integrals. Various numerical algorithms derived from the Radon inversion formulas or their equivalents are available for reconstructing single cross-sections of objects, see, e.g., [3], [5], [8], [9], [13], [16], [19].

This reduction of the dimension of the problem is not always possible for clinical and/or engineering reasons. For example, a device which collects data for a single cross-section at a time is not appropriate for the study of moving organs such as the beating heart. This is one of many motivations which led to the design of CT scanners using cone beams (the beam has the shape of a three-dimensional cone), e.g., the dynamic spatial reconstructor (DSR) [21] and the cardiovascular CT (CVCT) scanner [2].

There is no known closed-form inversion formula allowing the reconstruction of a three-dimensional object from cone-beam x-ray projection data with the vertex of the cone (e.g., x-ray source) describing a bounded curve. The formulas given by Gel'fand [4] and Kirillov [10] assume the vertex describes an unbounded curve. Other works related to this area can be found in [1], [6], [11], [12], [17], [20].

In this article we give an inversion formula for the reconstruction of a three-dimensional object from x-ray cone-beam where the vertex describes a bounded curve satisfying a set of weak conditions.

We shall assume that the density function f is a real integrable function on $\mathbf{R}^3$ whose support is contained in a compact set Ω. The inner product of two vectors $\alpha, \beta \in \mathbf{R}^3$ will be denoted by $\langle \alpha, \beta \rangle$. The integral of a function over a domain D in $\mathbf{R}^3$ is denoted by $\int_D$. The Fourier transform $\hat{f}$ of a function $f: \mathbf{R}^3 \to \mathbf{R}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^3} f(x)\, e^{-2i\pi\langle x, \xi\rangle}\, dx. \tag{1}$$

Then f is the inverse Fourier transform of $\hat{f}$, i.e.,

$$f(x) = \int_{\mathbf{R}^3} \hat{f}(\xi)\, e^{2i\pi\langle \xi, x\rangle}\, d\xi. \tag{2}$$

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Using spherical coordinates, relation (2) gives

$$f(x)=\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\cos\phi\int_0^{\infty}\rho^2\hat{f}(\rho\beta)\,e^{2i\pi\rho\langle\beta,x\rangle}\,d\rho\,d\phi\,d\theta, \tag{3}$$

where $\beta=(\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$.

A curve in $\mathbf{R}^3$ is a continuous function $\Phi:\Lambda\to\mathbf{R}^3$, where Λ is an interval of the real line $\mathbf{R}$. The unit sphere in $\mathbf{R}^3$ centered at the origin will be denoted by S.

Our inversion formula is valid if the vertex of the cone-beam describes a curve Φ satisfying the following conditions:

i) The curve is outside of the region Ω.

ii) The curve is bounded, continuous and almost everywhere differentiable.

iii) For all (x,β) in $\Omega\times S$, there exists λ in Λ, such that $\langle x,\beta\rangle=\langle\Phi(\lambda),\beta\rangle$ and $\langle\Phi'(\lambda),\beta\rangle\neq 0$.

The third condition of the curve means that, for any direction β, the plane orthogonal to β passing through a point $x\in\Omega$ must cut the curve at a point $\Phi(\lambda)$ for which $\langle\Phi'(\lambda),\beta\rangle\neq 0$. For any bounded object, a curve consisting of two circles such as that in Fig. 1 or some type of spiral curve will satisfy all the conditions on the curve required in the theorem.

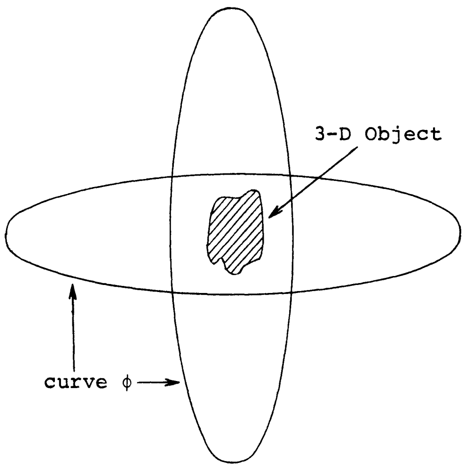

FIG. 1

Let $x_0\in\mathbf{R}^3$ and $\beta\in S$. The integral of f along the line of direction β and passing through x_0 can be written as $\int_{-\infty}^{\infty}f(x_0+t\beta)\,dt$. Since we assume that f has a support contained in Ω and the curve Φ is outside of Ω, it follows that the above line integral is not zero only along the half-lines or rays leaving $\Phi(\lambda)$ in a cone of vertex $\Phi(\lambda)$. In that case we have

$$\int_0^{\infty}f(\Phi(\lambda)+t\beta)\,dt=\int_{-\infty}^{\infty}f(\Phi(\lambda)+t\beta)\,dt. \tag{4}$$

DEFINITIONS. a) For $\alpha\in\mathbf{R}^3$, $\lambda\in\Lambda$, we define

$$g(\alpha,\lambda)=\int_0^{\infty}f(\Phi(\lambda)+t\alpha)\,dt=\frac{1}{|\alpha|}\int_0^{\infty}f\left(\Phi(\lambda)+t\frac{\alpha}{|\alpha|}\right)dt. \tag{5}$$

From the relation $\alpha=|\alpha|\beta$, where $\beta=\alpha/|\alpha|$, and the relation (4), it follows immediately that $g(\alpha,\lambda)$ is the product of $|\alpha|^{-1}$ with the integral of f along the line of direction $\alpha/|\alpha|$ and passing through the point $\Phi(\lambda)$. In medical imaging environment, $\Phi(\lambda)$ corresponds, for example, to a position of x-ray source and $g(\alpha,\lambda)$ can be obtained from the projection data along the ray leaving the point $\Phi(\lambda)$ in the direction $\alpha/|\alpha|$.

b) For $\xi\in\mathbf{R}^3$ and $\lambda\in\Lambda$, we define

$$G(\xi,\lambda)=\int_{\mathbf{R}^3} g(\alpha,\lambda)\, e^{-2i\pi\langle\alpha,\xi\rangle}\, d\alpha. \tag{6}$$

c) For each $\lambda\in\Lambda$, we define $\gamma_\lambda:\mathbf{R}^3\times(0,\infty)\to\mathbf{R}^3\times(0,\infty)$ by

$$\gamma_\lambda(\alpha,t)=\left(\Phi(\lambda)+t\alpha,\frac{1}{t}\right). \tag{7}$$

Note that γ_λ is bijective.

2. The inversion formula. We give only a formal derivation of the inversion formula in this section. A rigorous proof is given in the Appendix.

LEMMA. *For $\xi\in\mathbf{R}^3$ and $\lambda\in\Lambda$, we have*

$$G(\xi,\lambda)=\int_0^\infty \rho\hat{f}(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\, d\rho. \tag{8}$$

Proof. From the definitions (5) and (6) we obtain

$$G(\xi,\lambda)=\int_{\mathbf{R}^3}\int_0^\infty f(\Phi(\lambda)+t\alpha)\, e^{-2i\pi\langle\alpha,\xi\rangle}\, dt\, d\alpha.$$

Making a change of variables defined by γ_λ given in (7) and realizing that the Jacobian of the transformation γ_λ is equal to $-\rho$, we conclude that

$$\begin{aligned} G(\xi,\lambda)&=\int_{\mathbf{R}^3}\int_0^\infty \rho f(x)\, e^{-2i\pi\rho\langle x-\Phi(\lambda),\xi\rangle}\, d\rho\, dx\\ &=\int_0^\infty \rho\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\int_{\mathbf{R}^3} f(x)\, e^{-2i\pi\langle x,\rho\xi\rangle}\, dx\, d\rho. \end{aligned}$$

Hence (8) is proved.

THEOREM. *Let f be a real integrable function defined on $\mathbf{R}^3$ with support contained in a compact subset Ω. If Φ satisfies the curve conditions given in § 1, then for $x\in\Omega$,*

$$f(x)=\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\cos\phi\,\frac{1}{2i\pi\langle\Phi'(\lambda),\beta\rangle}\cdot\frac{\partial G(\beta,\lambda)}{\partial\lambda}\, d\phi\, d\theta, \tag{9}$$

where $\beta=(\cos\theta\cos\phi,\sin\theta\cos\phi,\sin\phi)$ and λ is such that $\langle\beta,x\rangle=\langle\beta,\Phi(\lambda)\rangle$ and $\langle\beta,\Phi'(\lambda)\rangle\neq 0$.

Proof. From the lemma, we obtain

$$\frac{\partial G}{\partial\lambda}(\xi,\lambda)=2i\pi\langle\Phi'(\lambda),\xi\rangle\int_0^\infty \rho^2\hat{f}(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\, d\rho. \tag{10}$$

The third condition on the curve allows us to conclude that for each fixed x and $\beta = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$, there exists λ in Λ, for which

$$\int_0^\infty \rho^2 \hat{f}(\rho\beta)\, e^{2i\pi\rho\langle x,\beta\rangle}\, d\rho = \frac{1}{2i\pi\langle\Phi'(\lambda), \xi\rangle} \cdot \frac{\partial G}{\partial\lambda}(\beta, \lambda).$$

Taking (3) into account, we conclude the relation (9).

Remarks. 1) For the curve and object like in Fig. 2, the third condition is not satisfied for every point of the object. However, it is met for every point in a region

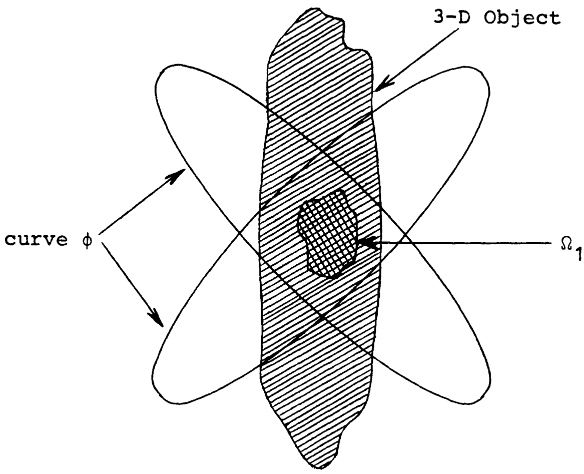

FIG. 2

containing Ω_1. According to the relation (9), the density function can be estimated in that region, provided that for each point of the curve, all the integrals of the function along the lines through the point are known in a cone containing the whole object.

2) The conditions on the curve seem to be natural from the following point of view: to get a good feeling about the shape of a three-dimensional object, usually we rotate the object about two perpendicular axes.

3) An inversion formula of the form (9) in the case where the function f is defined on $\mathbf{R}^n$ $(n \geqq 2)$, can be established using the arguments similar to those given above.

Appendix. There exist functions f for which $G(\xi, \lambda)$, see (8), and its derivative, see (10), do not exist as functions. The arguments given in the proof of the lemma were purely formal. The change of variables and the change of the order of integrations were applied even though the integrands were not necessarily absolutely integrable.

In order to give a rigorous proof, we consider $G(\xi, \lambda)$ and its derivative to be tempered distributions, i.e., continuous linear forms defined on the space $\mathcal{S}(\mathbf{R}^3)$ of rapidly decreasing C^∞-functions [14]. The value of a distribution T at a test function $\psi \in \mathcal{S}(\mathbf{R}^3)$ is denoted by $\langle T, \psi\rangle$. We recall that slowly increasing locally integrable functions can be considered as tempered distributions [14, p. 110], and if two functions define two equal distributions, then they are equal almost everywhere as functions [18, p. 80].

From (6) we see that G is the Fourier transform of g with respect to the variable α. The function g is slowly increasing; consequently it defines a tempered distribution.

Moreover its Fourier transform with respect to α is defined as

$$(11) \qquad \langle G(\xi,\lambda),\psi(\xi)\rangle = \langle g(\alpha,\lambda),\hat\psi(\alpha)\rangle = \int_{\mathbf{R}^3} g(\alpha,\lambda)\hat\psi(\alpha)\,d\alpha,$$

for all $\psi \in \mathscr{S}(\mathbf{R}^3)$, see [14, p. 118].

For $k=1,2$, we define

$$\int_0^\infty \rho^k \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho$$

as

$$\lim_{m\to\infty}\int_0^m \rho^k \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho$$

in the distribution sense, i.e., for each $\psi\in\mathscr{S}(\mathbf{R}^3)$,

$$(12) \quad \left\langle \int_0^\infty \rho^k \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho,\ \psi(\xi)\right\rangle = \lim_{m\to\infty}\left\langle \int_0^m \rho^k \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho,\ \psi(\xi)\right\rangle.$$

Here $\langle h(\xi),\psi(\xi)\rangle$ is $\int_{\mathbf{R}^3} h(\xi)\psi(\xi)\,d\xi$ whenever h is a function. The following proposition and its proof justify the above definitions.

PROPOSITION 1. *For $\psi\in\mathscr{S}(\mathbf{R}^3)$ and $\lambda\in\Lambda$,*

$$(13) \qquad \left\langle \int_0^\infty \rho \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho,\ \psi(\xi)\right\rangle = \int_0^\infty \rho\int_\Omega f(x)\hat\psi(\rho(x-\Phi(\lambda)))\,dx\,d\rho$$

and

$$(14) \qquad \frac{\partial}{\partial\lambda}\int_0^\infty \rho \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho = 2i\pi\langle\Phi'(\lambda),\xi\rangle\int_0^\infty \rho^2 \hat f(\rho\xi)\, e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho.$$

Proof. We know that $\hat\psi$ is rapidly decreasing [14, p. 116]. Since there exists $a>0$ such that $|x-\Phi(\lambda)|\geqq a$ for all $x\in\Omega$ and $\lambda\in\Lambda$, it follows that

$$(15) \quad \int_0^\infty\int_\Omega \rho|f(x)\hat\psi(\rho(x-\Phi(\lambda)))|\,dx\,d\rho \leqq C\int_\Omega |f(x)|\,dx\int_0^\infty \frac{\rho}{(1+a^2\rho^2)^n}\,d\rho,$$

for some constant $n\geqq 2$ and $C>0$. By the Lebesgue dominated convergence theorem and the Fubini theorem, we conclude

$$\int_0^\infty \rho\int_\Omega f(x)\hat\psi(\rho(x-\Phi(\lambda)))\,dx\,d\rho = \lim_{m\to\infty}\int_0^m \rho\int_\Omega f(x)\hat\psi(\rho(x-\Phi(\lambda)))\,dx\,d\rho.$$

Moreover we have,

$$\begin{aligned}
&\int_0^m \rho\int_\Omega f(x)\hat\psi(\rho(x-\Phi(\lambda)))\,dx\,d\rho\\
&\quad=\int_0^m \rho\int_\Omega f(x)\int_{\mathbf{R}^3}\psi(\xi)\,e^{-2i\pi\rho\langle x-\Phi(\lambda),\xi\rangle}\,d\xi\,dx\,d\rho\\
&\quad=\int_{\mathbf{R}^3}\psi(\xi)\int_0^m \rho\hat f(\rho\xi)\,e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho\,d\xi.
\end{aligned}$$

Taking definition (12) into account, we obtain the relation (13).

The relation (15) allows us to write

$$\frac{\partial}{\partial\lambda}\int_0^\infty \rho\int_\Omega f(x)\hat{\psi}(\rho(x-\Phi(\lambda)))\,dx\,d\rho$$
$$=\int_0^\infty \rho\int_\Omega f(x)\frac{\partial}{\partial\lambda}\hat{\psi}(\rho(x-\Phi(\lambda)))\,dx\,d\rho$$
$$=\lim_{m\to\infty}\int_0^m \rho^2\int_\Omega f(x)\int_{\mathbf{R}^3}\psi(\xi)2i\pi\langle\Phi'(\lambda),\xi\rangle e^{-2i\pi\rho\langle x-\Phi(\lambda),\xi\rangle}\,d\xi\,dx\,d\rho$$
$$=\lim_{m\to\infty}\int_{\mathbf{R}^3}\psi(\xi)2i\pi\langle\Phi'(\lambda),\xi\rangle\int_0^m \rho^2\hat{f}(\rho\xi)\,e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho\,d\xi.$$

Using the definition (12) and the relation (13) we can conclude (14).

PROPOSITION 2. *As tempered distributions, we have*

$$G(\xi,\lambda)=\int_0^\infty \rho\hat{f}(\rho\xi)\,e^{2i\pi\rho\langle\Phi(\lambda),\xi\rangle}\,d\rho. \tag{16}$$

Proof. From the definitions (11) of G and (5) of g, we obtain

$$\langle G(\xi,\lambda),\psi(\xi)\rangle=\int_{\mathbf{R}^3}\int_0^\infty f(\Phi(\lambda)+t\alpha)\hat{\psi}(\alpha)\,dt\,d\alpha.$$

The integrand $f(\Phi(\lambda)+t\alpha)\hat{\psi}(\alpha)$ is absolutely integrable since $\hat{\psi}$ is rapidly decreasing and g as defined in (5), is slowly increasing as a function of α. After making a change of variables defined by γ_λ, we have

$$\langle G(\xi,\lambda),\psi(\xi)\rangle=\int_0^\infty\int_{\mathbf{R}^3}\rho f(x)\hat{\psi}(\rho(x-\Phi(\lambda)))\,dx\,d\rho.$$

The conclusion follows from the relation (13).

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