Tomography

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.
  
  - Projections must be collected at every angle $\theta$ and displacement $r$.
  
  - Forward projections $p_{\theta}(r)$ are known as a Radon transform.

- Objective: reverse this process to form the original image $f(x, y)$.
  
  - Fourier Slice Theorem is the basis of inverse
  
  - Inverse can be computed using convolution back projection (CBP)
Medical Imaging Modalities

• Anatomical Imaging Modalities
  – Chest X-ray
  – Computed Tomography (CT)
  – Magnetic Resonance Imaging (MRI)

• Functional Imaging Modalities
  – Signal Photon Emission Tomography (SPECT)
  – Positron Emission Tomography (PET)
  – Functional Magnetic Resonance Imaging (fMRI)
Multislice Helical Scan CT

- Multislice CT has a cone-beam structure
Example: CT Scan

- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan
Photon Attenuation

$x$ - depth into material measured in cm

$Y_x$ - Number of photons at depth $x$

$\lambda_x = E[Y_x]$

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x} \lambda_x^k}{k!}.$$ 

- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.
The attenuation of photons obeys the following equation
\[
\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x
\]
where \(\mu(x)\) is the density in units of cm\(^{-1}\).

- The solution to this equation is given by
  \[
  \lambda_x = \lambda_0 e^{-\int_0^x \mu(t)dt}
  \]
- So we see that
  \[
  \int_0^x \mu(t)dt = -\log\left(\frac{\lambda_x}{\lambda_0}\right)
  \approx -\log\left(\frac{Y_x}{\lambda_0}\right)
  \]
A commonly used estimate of the projection integral is

$$
\int_0^T \mu(t) dt \approx - \log \left( \frac{Y_T}{\lambda_0} \right)
$$

where:

- $\lambda_0$ is the dosage
- $Y_T$ is the photon count at the detector
Positron Emission Tomography (PET)

- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements

\[ E[y_i] = \sum_j A_{ij} x_j \]
Example: PET/CT Scan

- Generally low space/time resolution
- Little anatomical detail $\Rightarrow$ couple with CT
- Can detect disease
Coordinate Rotation

- Define the counter-clockwise rotation matrix

\[ A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]

- Define the new coordinate system \((r, z)\)

\[
\begin{bmatrix} x \\ y \end{bmatrix} = A_\theta \begin{bmatrix} r \\ z \end{bmatrix}
\]

- Geometric interpretation

- Inverse transformation

\[
\begin{bmatrix} r \\ z \end{bmatrix} = A_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}
\]
Integration Along Projections

• Consider the function $f(x, y)$.

• We compute projections by integrating along $z$ for each $r$.

• The projection integral for each $r$ and $\theta$ is given by

$$p_\theta(r) = \int_{-\infty}^{\infty} f \left(A_\theta \left[ \begin{array}{c} r \\ z \end{array} \right] \right) dz$$

$$= \int_{-\infty}^{\infty} f \left( r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta) \right) dz$$
The Radon Transform

- The Radon transform of the function $f(x, y)$ is defined as
  \[ p_{\theta}(r) = \int_{-\infty}^{\infty} f \left( r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta) \right) dz \]

- The geometric interpretation is
  
  Notice that the projection corresponding to $r = 0$ goes through the point $(x, y) = (0, 0)$. 

The Fourier Slice Theorem

• Let

\[ P_\theta(\rho) = CTFT \{ p_\theta(r) \} \]
\[ F(u, v) = CSFT \{ f(x, y) \} \]

Then

\[ P_\theta(\rho) = F(\rho \cos(\theta), \rho \sin(\theta)) \]

• \( P_\theta(\rho) \) is \( F(u, v) \) in polar coordinates!
Proof of the Fourier Slice Theorem

- By definition

\[ p_{\theta}(r) = \int_{-\infty}^{\infty} f \left( A_{\theta} \left[ \begin{array}{c} r \\ z \end{array} \right] \right) \, dz \]

- The CTFT of \( p_{\theta}(r) \) is then given by

\[
P_{\theta}(\rho) = \int_{-\infty}^{\infty} p_{\theta}(r) e^{-j2\pi\rho r} \, dr
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f \left( A_{\theta} \left[ \begin{array}{c} r \\ z \end{array} \right] \right) \, dz \right] e^{-j2\pi\rho r} \, dr
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \left( A_{\theta} \left[ \begin{array}{c} r \\ z \end{array} \right] \right) e^{-j2\pi\rho r} \, dz \, dr
\]

- We next make the change of variables

\[
\left[ \begin{array}{c} r \\ z \end{array} \right] = A_{-\theta} \left[ \begin{array}{c} x \\ y \end{array} \right].
\]

Notice that the Jacobian is \( |A_{\theta}| = 1 \), and that \( r = x \cos(\theta) + y \sin(\theta) \). This results in

\[
P_{\theta}(\rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \rho [x \cos(\theta) + y \sin(\theta)]} \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi [x \rho \cos(\theta) + y \rho \sin(\theta)]} \, dx \, dy
\]

\[
= F(\rho \cos(\theta), \rho \sin(\theta))
\]
Alternative Proof of the Fourier Slice Theorem

• First let $\theta = 0$, then

$$p_0(r) = \int_{-\infty}^{\infty} f(r, y) \, dy$$

Then

$$P_0(\rho) = \int_{-\infty}^{\infty} p_0(r) e^{-2\pi jr \rho} \, dr$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(r, y) \, dy \right] e^{-2\pi jr \rho} \, dr$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, y) e^{-2\pi j(r\rho+y0)} \, dr \, dy$$

$$= F(\rho, 0)$$

• By rotation property of CSFT, it must hold for any $\theta$. 
Inverse Radon Transform

- Physical systems measure $p_\theta(r)$.
- From these, we compute $P_\theta(\rho) = CTFT\{p_\theta(r)\}$.

- Next we take an inverse CSFT to form $f(x, y)$.

Problem: This requires polar to rectangular conversion.

Solution: Convolution backprojection
Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of $F(u, v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du \; dv = |\rho|d\theta \; d\rho$$

- This results in the expression

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{2\pi j(xu+yv)} \; du \; dv$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\pi} P_\theta(\rho)e^{2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))} |\rho|d\theta \; d\rho$$

$$= \int_{0}^{\pi} \left[ \int_{-\infty}^{\infty} |\rho|P_\theta(\rho)e^{2\pi j\rho(x\cos(\theta)+y\sin(\theta))} \; d\rho \right] d\theta$$

- Then $g(t)$ is given by

$$g_\theta(t) = \int_{-\infty}^{\infty} |\rho|P_\theta(\rho)e^{2\pi j\rho t} \; d\rho$$

$$= CTFT^{-1}\{|\rho|P_\theta(\rho)\}$$

$$= h(t) * p_\theta(r)$$
where \( h(t) = CTFT^{-1} \{ |\rho| \} \), and

\[
f(x, y) = \int_{0}^{\pi} g_{\theta}(x \cos(\theta) + y \sin(\theta)) \, d\theta
\]
Summary of CBP Algorithm

1. Measure projections $p_\theta(r)$.
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.
3. Back project filtered projections

$$f(x, y) = \int_0^\pi g_\theta(x \cos(\theta) + y \sin(\theta)) \, d\theta$$
A Closer Look at Projection Filter

1. At each angle, projections are filtered.

\[ g_\theta(r) = h(r) \ast p_\theta(r) \]

2. The frequency response of the filter is given by

\[ H(\rho) = |\rho| \]

3. But real filters must be bandlimited to \( |\rho| \leq f_c \) for some cut-off frequency \( f_c \).

So

\[ H(\rho) = f_c \left[ \text{rect} \left( \frac{f}{2f_c} \right) - \Lambda \left( \frac{f}{f_c} \right) \right] \]

\[ h(r) = f_c^2 \left[ 2\text{sinc}(2f_c r) - \text{sinc}^2(f_c r) \right] \]
A Closer Look at Back Projection

- Back Projection function is

\[ f(x, y) = \int_{0}^{\pi} b_{\theta}(x, y) \, d\theta \]

where

\[ b_{\theta}(x, y) = g_{\theta}(x \cos(\theta) + y \sin(\theta)) \]

- Consider the set of points \((x, y)\) such that

\[ r = x \cos(\theta) + y \sin(\theta) \]

This set looks like

- Along this line \(b_{\theta}(x, y) = g_{\theta}(r)\).
Back Projection Continued

- For each angle $\theta$ back projection is constant along lines of angle $\theta$ and takes on value $g_\theta(r)$.

- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to $\pi$.

\[
f(x, y) = \int_0^\pi b_\theta (x, y) \, d\theta
\]

\[
\approx \frac{\pi}{M} \sum_{m=0}^{M-1} \frac{b_{m\pi}}{M} (x, y)
\]

- Back projection “smears” values of $g(r)$ back over image, and then adds smeared images for each angle.