

## Random Variables

- Let  $X$  be a random variable on  $\mathbb{R}$ , then
  - $X$  is usually denoted by an upper case letter.
  - The cumulative distribution function is given by

$$P\{X \leq x\} = F_X(x)$$

- If the probability density function exists, it is given by

$$p_X(x) = \frac{dF_X(x)}{dx}$$

so that

$$\begin{aligned} P\{x_1 < X \leq x_2\} &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} p_X(\tau) d\tau \end{aligned}$$

- The expectation of  $X$  is given by

$$E[X] = \int_{-\infty}^{\infty} \tau p_X(\tau) d\tau$$

or more precisely by the Riemann-Stieltjes integral

$$E[X] = \int_{-\infty}^{\infty} \tau dF_X(\tau)$$

if it exists.

## Deterministic versus Random

- Let  $X$  and  $Z$  be random variables, and let  $f(\cdot)$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$ 
  - Is  $Y = f(X)$  a random variable?
  - Is  $\mu = E[X]$  a random variable?
  - Is  $\hat{X} = E[X|Z]$  a random variable?

## Properties of Expectation

- Expectation is linear

$$E[X + Y] = E[X] + E[Y]$$

- What is  $E[E[X|Y]]$  equal to?

$$E[E[X|Y]] = E[X]$$

- What is  $E[X|X, Y]$  equal to?

$$E[X|X, Y] = X$$

- When  $X$ ,  $Y$ , and  $Z$  are (jointly) Gaussian

$$E[X|Y, Z] = aY + bZ + c$$

for some scalar values  $a$ ,  $b$ , and  $c$ .

## 2-D Discrete Space Random Processes

- Notation

- $X_s$  is a pixel at position  $s = (s_1, s_2) \in \mathcal{Z}^2$
- $S$  denotes the set of 2-D Lattice points where  $S \subset \mathcal{Z}^2$

- Definitions

- Mean  $\mu_s = E[X_s]$
- Autocorrelation  $R_{sr} = E[X_s X_r]$
- Autocovariance  $C_{sr} = E[(X_s - \mu_s)(X_r - \mu_r)]$
- A process is said to be **second order** if  $E[X_s]$  and  $E[X_s X_r]$  exist for all  $s \in S$  and  $r \in S$ .
- A second order random process is said to be **wide sense stationary** if for all  $s \in \mathcal{Z}^2$

$$\mu_s = \mu_{(0,0)}$$

$$C_{r,r+s} = C_{(0,0),s}$$

## 2-D Power Spectral Density

Let  $X_s$  be a zero mean wide sense stationary random process.

Define

$$\hat{X}_N(e^{j\mu}, e^{j\nu}) = \sum_{m=-N}^N \sum_{n=-N}^N X_{(m,n)} e^{-j(m\mu+n\nu)}$$

- Then the power spectrum (i.e. energy spectrum per unit sample) is

$$\frac{1}{(2N+1)^2} \left| \hat{X}_N(e^{j\mu}, e^{j\nu}) \right|^2$$

The following limit does not converge!!

$$\lim_{N \rightarrow \infty} \frac{1}{(2N+1)^2} \left| \hat{X}_N(e^{j\mu}, e^{j\nu}) \right|^2$$

Intuition - The spectral estimate remains noisy as the window size increases.

## Definition of Power Spectral Density

- Definition of **Power Spectral Density**

$$S_x(e^{j\mu}, e^{j\nu}) \triangleq \lim_{N \rightarrow \infty} \frac{1}{(2N+1)^2} E \left[ \left| \hat{X}_N(e^{j\mu}, e^{j\nu}) \right|^2 \right]$$

Expectation removes the noise.

## Weiner-Khintchine Theorem

- For a wide sense stationary random process, the power spectral density equals the Fourier transform of the auto-correlation

$$S_x(e^{j\mu}, e^{j\nu}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m, n) e^{-j(m\mu + n\nu)}$$

where

$$R(m, n) = E[X_{(0,0)} X_{(m,n)}]$$

## Estimating Power Spectral Density

- For simplicity, consider 1-D case

$$S_x(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{N} E \left[ \left| \hat{X}_N(e^{j\omega}) \right|^2 \right]$$

How do we compute the required expectation?

- Answer: The law of large numbers (averaging)

$$E[Z] = \lim_{N \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} Z_k$$

where  $Z_k$  are independent and identically distributed (i.i.d.).



## Computing Power Spectrum from Block Average

- Let  $X_n$  be signal with  $0 \leq n < NK$ .
- Break  $X_n$  into  $K$  parts, each of length  $N$ .

$$Y_n^{(k)} = X_{kN+n}$$

where  $0 \leq k < K$  and  $0 \leq n < N$ .

- Compute DTFT of  $Y_n^{(k)}$

$$\hat{Y}^{(k)}(e^{j\omega}) = \sum_{n=0}^{N-1} Y_n^{(k)} e^{j\omega n}$$

- Average power spectrum estimates for each block

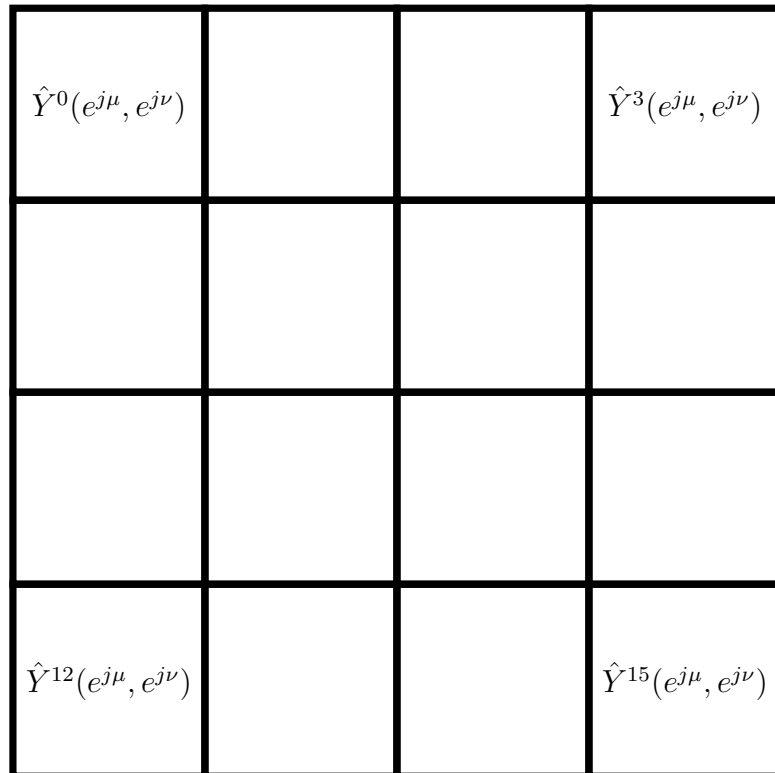
$$\begin{aligned} S_x(e^{j\omega}) &= \frac{1}{N} E \left[ \left| \hat{Y}^{(k)}(e^{j\omega}) \right|^2 \right] \\ &\approx \frac{1}{N} \left[ \frac{1}{K} \sum_{k=0}^{K-1} \left| \hat{Y}^{(k)}(e^{j\omega}) \right|^2 \right] \end{aligned}$$

- So we have that

$$S_x(e^{j\omega}) \approx \frac{1}{NK} \sum_{k=0}^{K-1} \left| \hat{Y}^{(k)}(e^{j\omega}) \right|^2$$

## Block Averaging in 2-D

- Break image into  $K$  regions each with  $N$  pixels



- For each block, compute  $\hat{Y}^{(k)}(e^{j\mu}, e^{j\nu})$
- Average blocks to form power spectrum estimate

$$S_x(e^{j\mu}, e^{j\nu}) \cong \frac{1}{NK} \sum_{k=0}^{K-1} \left| \hat{Y}^{(k)}(e^{j\mu}, e^{j\nu}) \right|^2$$