Nonlinear Filtering

- Linear filters
  - Tend to blur edges and other image detail.
  - Perform poorly with non-Gaussian noise.
  - Result from Gaussian image and noise assumptions.
  - Images are not Gaussian.

- Nonlinear filter
  - Can preserve edges
  - Very effective at removing impulsive noise
  - Result from non-Gaussian image and noise assumptions.
  - Can be difficult to design.
Linear Filters

• Definition: A system \( y = T[x] \) is said to be linear if for all \( \alpha, \beta \in \mathbb{R} \)

\[
\alpha y_1 + \beta y_2 = T[\alpha x_1 + \beta x_2]
\]

where \( y_1 = T[x_1] \) and \( y_2 = T[x_2] \).

• Any filter of the form

\[
y_s = \sum_r h_{s,r} x_r
\]
Homogeneous Filters

- Definition: A filter $y = T[x]$ is said to be homogeneous if for all $\alpha \in \mathbb{R}$
  $$\alpha y = T[\alpha x]$$

- This is much weaker than linearity.
- Homogeneity is a natural condition for scale invariant systems.
Median Filter

• Let $W$ be a window with an odd number of points.
• Then the median filter is given by
  \[ y_s = \text{median} \{ x_{s+r} : r \in W \} \]
• Is the median filter:
  – Linear?
  – Homogeneous?

• Consider the 1-D median filter with a 3-point window.

<table>
<thead>
<tr>
<th>x(m)</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1,000</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y(m)</td>
<td>?</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>
**Median Filter: Optimization Viewpoint**

- Consider the median filter
  \[ y_s = \text{median} \{ x_{s+r} : r \in W \} \]
  and consider the following functional.
  \[ F_s(\theta) \triangleq \sum_{r \in W} |\theta - x_{s+r}| \]

- Then \( y_s \) solves the following optimization equation.
  \[ y_s = \arg \min_{\theta} F_s(\theta) \]

- Differentiating, we have
  \[
  \frac{d}{d\theta} F(\theta) = \frac{d}{d\theta} \sum_{r \in W} |\theta - x_{s+r}|
  = \sum_{r \in W} \text{sign}(\theta - x_{s+r})
  \triangleq f(\theta)
  \]

  This expression only holds for \( \theta \neq x_{s+r} \) for all \( r \in W \).

- So the solution falls at \( \theta = x_{s^*} \) such that
  \[
  0 = \sum_{r \in W} \text{sign}(\theta - x_{s+r})
  \]
  \( r \neq (s^* - s) \)
Example: Median Filter Function

- Consider a 1-D median filter
  - Three point window of $W = \{-1, 0, 1\}$
  - Inputs $[x(n-1), x(n), x(n+1)] = [-2, -4, 5]$.

\[
F(\theta) = \sum_{k=-1}^{1} |\theta - x_{n+k}|
\]
Example: Derivative of Median Filter Function

- Consider a 1-D median filter
  - Three point window of $W = \{-1, 0, 1\}$
  - Inputs $[x(n-1), x(n), x(n+1)] = [-2, -4, 5]$.

$$f(\theta) = \sum_{k=-1}^{1} \text{sign}(\theta - x_{n+k})$$
Problem with an Even Number of Points

• Consider a 1-D median filter
  – Four point window of \( W = \{-1, 0, 1, 2\} \)
  – Inputs \([x(n-1), x(n), x(n+1), x(n+2)] = [-2, -4, 5, 6]\).  

• Solution is not unique.

\[
F(\theta) = \sum_{k=-1}^{2} |\theta - x_{n+k}|
\]

Median Function \( F(\theta) \)
Weighted Median Filter

- Defined the functional

\[ F(\theta) \triangleq \sum_{r \in W} a_r |\theta - x_{s+r}| \]

where \( a_r \) are weights assigned to each point in the window \( W \).

- Weighted median is computed by

\[ y_s = \arg \min_{\theta} \sum_{r \in W} a_r |\theta - x_{s+r}| \]

- Differentiating, we have

\[ \frac{d}{d \theta} F(\theta) = \frac{d}{d \theta} \sum_{r \in W} a_r |\theta - x_{s+r}| \]

\[ = \sum_{r \in W} a_r \text{sign}(\theta - x_{s+r}) \]

\[ \triangleq f(\theta) \]

This expression only holds for \( \theta \neq x_r \) for all \( r \in W \).

- Need to find \( s^* \) such that \( f(\theta) \) is “nearly” zero.
Example: Weighted Median Filter Function

- Consider a 1-D median filter
  
  – Five point window of $W = \{-2, -1, 0, 1, 2\}$
  – Inputs $[x(n-2), \cdots, x(n+2)] = [6, -2, -4, 5, -1]$.
  – Weights $[a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1]$.

$$F(\theta) = \frac{1}{k=-1} a(k)|\theta - x_{n+k}|$$
Example: Derivative of Median Filter Function

- Consider a 1-D median filter
  - Five point window of \( W = \{-2, -1, 0, 1, 2\} \)
  - Inputs \([x(n-2), \cdots, x(n+2)] = [6, -2, -4, 5, -1]\).
  - Weights \([a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1]\).

\[
f(\theta) = \sum_{k=-1}^{1} a(k) \text{sign}(\theta - x_{n+k})
\]
Computation of Weighted Median

1. Sort points in window.
   - Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(p)}$ be the sorted values.
   - These values are known as order statistics.
   - Let $a_{(1)}, a_{(2)}, \cdots, a_{(p)}$ be the corresponding weights.

2. Find $i^*$ such that the following equations hold

\[
\begin{align*}
    a_{i^*} + \sum_{i=1}^{i^*-1} a(i) & \geq \sum_{i=i^*+1}^{p} a(i) \\
    \sum_{i=1}^{i^*-1} a(i) & \leq \sum_{i=i^*+1}^{p} a(i) + a_{i^*}
\end{align*}
\]

3. Then the value $x_{(i^*)}$ is the weighted median value.
Comments on Weighted Median Filter

- Weights may be adjusted to yield the “best” filter.
- Largest and smallest values are ignored.
- Same as median filter for $a_r = 1$. 