Filtered Random Processes

- Consider the 2-D linear system

$$Y(m, n) = h(m, n) * X(m, n)$$

where $X(m, n)$ is a 2-D wide sense stationary random process.

- It may be easily shown that

$$R_y(m, n) = h(m, n) * h(-m, -n) * R_x(m, n)$$

$$S_y(e^{j\mu}, e^{j\nu}) = \left| H(e^{j\mu}, e^{j\nu}) \right|^2 S_x(e^{j\mu}, e^{j\nu})$$

2D Gaussian White Noise

- Definition:

  - $X(m, n)$ - independent identically distributed (i.i.d.) Gaussian random variables with distribution $N(0, 1)$.

- Then

  - $X(m, n)$ is wide sense stationary with

$$\mu(m, n) = E[X(m, n)] = 0$$

$$R_x(k, l) = E[X(0, 0)X(k, l)] = \delta(k, l)$$

$$S_x(e^{j\mu}, e^{j\nu}) = DSFT\{R_x(k, l)\}$$

$$= 1$$

Filtered White Noise

- Definitions:

  - $X(m, n)$ - independent identically distributed (i.i.d.) Gaussian random variables with distribution $N(0, 1)$.
  - $Y(m, n) = h(m, n) * X(m, n)$.

- Then

  - $Y(m, n)$ is wide sense stationary with

$$
\begin{aligned}
S_y(e^{j\mu}, e^{j\nu}) &= \left| H(e^{j\mu}, e^{j\nu}) \right|^2 S_x(e^{j\mu}, e^{j\nu}) \\
&= \left| H(e^{j\mu}, e^{j\nu}) \right|^2 \cdot 1 \\
R_y(k, l) &= h(m, n) * h(-m, -n) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m, n) h(m + k, n + l)
\end{aligned}
$$

  - $R_y(k, l)$ is the autocorrelation of $h(m, n)$ with itself.

Causal Prediction

- Y_s is a 2-D wide sense stationary zero mean Gaussian random process.

- Define

  - The past values are $Y_{<s} = \{Y_r : r < s\}$.

  - The minimum mean squared error (MMSE) predictor of Y_s given the past is

  $$\hat{Y}_s = E[Y_s|Y_{<s}]$$

  - The prediction error is $X_s = Y_s - \hat{Y}_s$.

Properties of Causal Predictors

- Fact 1: (WLOG, assume $r < s$.)

$$
\begin{aligned}
E[X_s X_r] &= E[E[X_s X_r | Y_{<s}]] \\
&= E[E[(Y_s - \hat{Y}_s)(Y_r - \hat{Y}_r)|Y_{<s}]] \\
&= E[E[(Y_s - \hat{Y}_s)|Y_{<s}](Y_r - \hat{Y}_r)] \\
&= E[(E[Y_s|Y_{<s}] - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
&= E[(\hat{Y}_s - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
&= E[0(Y_r - \hat{Y}_r)] = 0
\end{aligned}
$$

- Fact 2: $\sigma^2 \triangleq E[X_s^2]$ is the prediction variance.

- Fact 3: The predictor must be space invariant and linear.

$$
\hat{Y}_s = \sum_{r>(0,0)} h_r Y_{s-r}
$$

Autoregressive (AR) Processes

- Definitions:

  - Y_s - 2-D wide sense stationary zero mean Gaussian random process.
  - h_s - MMSE linear predictor for Y_s.
  - $X_s = Y_s - h_s * Y_s$ - predictor error.

- If h_s is FIR, then Y_s is known as an autoregressive (AR) process.

Properites of AR Processes

- Remember that

$$X_s = Y_s - h_s * Y_s$$

- Then

  - We know that

  $$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

  - Since X_s is white noise,

  $$R_x(s) = \sigma^2 \delta(s)$$

  $$S_x(e^{j\mu}, e^{j\nu}) = \sigma^2$$

  - So the power spectrum of Y_s is given by

  $$S_y(e^{j\mu}, e^{j\nu}) = \frac{\sigma^2}{|1 - H(e^{j\mu}, e^{j\nu})|^2}$$

Spectral Estimate Using AR Processes

- Compute MMSE linear predictor $\hat{h}_s$ for Y_s.

- Compute the prediction variance

$$\hat{\sigma}^2 = \frac{1}{|S|} \sum_{s \in S} |Y_s - h_s * Y_s|^2$$

  where S is a finite set of points in plain, and $|S|$ is the number of points in S.

- Estimate the power spectrum

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\hat{\sigma}^2}{\left|1 - \hat{H}(e^{j\mu}, e^{j\nu})\right|^2}$$

- Can produce a more accurate estimate of the power spectrum.

Generating AR Processes

- Select a causal prediction filter h_s.

- Apply IIR filter to white noise random process X_s

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Y_s is sometimes referred to as a white noise driven process.

- Do linear FIR prediction filters $\hat{h}_s$ always form a stable IIR filter?

    - In 1-D, yes.

    - In 2-D, not always!

- Other problems:

    - Causal ordering of points may cause asymmetric artifacts in results.

    - Complexity increases rapidly with IIR filter order P.