Notes on Fourier Optics

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1 Notation

The following is notation used this the document

λ - wavelength of light

$[v_x, v_y, v_z]$ - spatial frequency in (cycles/unit length)

$[k_x, k_y, k_z] = 2\pi[v_x, v_y, v_z]$ - spatial frequency in (radians/unit length)

$k = \|[k_x, k_y, k_z]\| = \frac{2\pi}{\lambda}$ - spatial frequency in (radians/unit length)

$\|[v_x, v_y, v_z]\| = \frac{1}{\lambda}$ - spatial frequency in (cycles/unit length)

Define the Fourier transform and its inverse as

$$\begin{aligned} G(v_x, v_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\exp\{-j2\pi(v_x x + v_y y)\}dxdy \\ g(x, y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(v_x, v_y)\exp\{j2\pi(v_x x + v_y y)\}dv_x dv_y \ . \end{aligned}$$

2 Derivation of Fresnel Transfer Function

Consider an optical plane wave at a single wavelength λ, then its electric field has the form

$$E(x, y, z) = \exp\{j2\pi(v_x x + v_y y + v_z z)\} \ .$$

Further consider two parallel planes one at position $z = 0$ and another at position $z = d$. Then in the presence of a plane waves the electric filed on the two planes is given by

$$\begin{aligned} u_0(x, y) &= E(x, y, 0) = \exp\{j2\pi(v_x x + v_y y)\} \\ u_d(x, y) &= E(x, y, d) = H(v_x, v_y)\exp\{j2\pi(v_x x + v_y y)\} \ , \end{aligned}$$

where H is a complex phase of the wave on the second plane given by $H(v_x, v_y) = \exp\{j2\pi v_z d\}$. Using the Fresnel approximation, the complex phase $H(v_x, v_y)$ is given by

$$\begin{aligned} H(v_x, v_y) &= \exp\{j2\pi v_z d\} \\ &= \exp\left\{j2\pi\left(\frac{1}{\lambda^2} - v_x^2 - v_y^2\right)^{1/2} d\right\} \\ &= \exp\left\{j2\pi\frac{d}{\lambda}\left(1 - \lambda^2[v_x^2 + v_y^2]\right)^{1/2}\right\} \\ &\approx \exp\left\{j2\pi\frac{d}{\lambda}\left(1 - \frac{\lambda^2[v_x^2 + v_y^2]}{2}\right)\right\} \\ &= \exp\left\{j2\pi\frac{d}{\lambda}\right\}\exp\left\{-j\pi\lambda d(v_x^2 + v_y^2)\right\} \ . \end{aligned}$$

Now we now know that when u_0 and u_d are plane waves with 2D frequency (v_x, v_y), then

$$u_d(x,y) = H(v_x, v_y) u_0(x,y) \ .$$

So therefore, $H(v_x, v_y)$ represents the 2D transfer function from the first plane to the second. More generally, if we let $U_d(v_x, v_y)$ and $U_0(v_x, v_y)$ be the 2D Fourier transforms of the more general functions $u_d(x,y)$ and $u_0(x,y)$, then we can use the Fresnel approximate to represent the relationship between the complex signals on the two planes.

Using the Fresnel approximation, the relationship between the complex signals on the two planes is given by

$$U_d(v_x, v_y) = H(v_x, v_y) U_0(v_x, v_y)$$

where

$$H(v_x, v_y) = \exp\left\{ j2\pi \frac{d}{\lambda} \right\} \exp\left\{ -j\pi\lambda d (v_x^2 + v_y^2) \right\} \ ,$$

when $\|[v_x, v_y]\| << \frac{1}{\lambda}$.

3 Derivation of Fresnel Convolution

If we take the inverse 2D Fourier transform of $H(v_x, v_y)$, then we get the impulse response associated with the Fresnel approximation given by

$$h(x,y) = \frac{1}{j\lambda d} \exp\left\{ j2\pi \frac{d}{\lambda} \right\} \exp\left\{ j\pi \frac{x^2 + y^2}{\lambda d} \right\} \ . \tag{1}$$

(See Section 5 below for details.)

Using this relationship, we can represent the relationship between the signals on the two planes with a convolution.

Using the Fresnel approximation in the space domain, we have that

$$\begin{aligned} u_d(x,y) &= u_0(x,y) * h(x,y) \\ &= h_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0(x', y') \exp\left\{ j\pi \frac{(x - x')^2 + (y - y')^2}{\lambda d} \right\} dx' dy' \ . \end{aligned}$$

where

$$h_0 = \frac{1}{j\lambda d} \exp\left(j2\pi \frac{d}{\lambda} \right) \ .$$

4 The Fraunhofer approximation

In the case that $x'^2 + y'^2$ is sufficiently small, then the Fraunhofer approximation holds.

$$
\begin{aligned}
u_d(x,y) &= h_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x',y') \exp\left\{ j\pi \frac{(x-x')^2+(y-y')^2}{\lambda d} \right\} dx'dy' \\
&= h_0 \exp\left\{ j\pi \frac{x^2+y^2}{\lambda d} \right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x',y') \exp\left\{ j\pi \frac{x'^2+y'^2}{\lambda d} \right\} \exp\left\{ -j2\pi \frac{xx'+yy'}{\lambda d} \right\} dx'dy' \\
&\approx h_0 \exp\left\{ j\pi \frac{x^2+y^2}{\lambda d} \right\} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x',y') \exp\left\{ -j2\pi \frac{xx'+yy'}{\lambda d} \right\} dx'dy' \\
&= h_0 \exp\left\{ j\pi \frac{x^2+y^2}{\lambda d} \right\} F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d} \right)
\end{aligned}
$$

where F is the Fourier transform given by

$$F(v_x, v_y) = \mathcal{F}\{u_0(x,y)\}$$

So relationship resulting from the Fraunhofer approximation is

$$u_d(x,y) = h_0 \exp\left\{ j\pi \frac{x^2+y^2}{\lambda d} \right\} F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d} \right)$$

where

$$F(v_x, v_y) = \mathcal{F}\{u_0(x,y)\} \ .$$

5 Fourier Transform Pairs

Define the 1D Fourier transform as

$$F(v) = \int_{-\infty}^{\infty} f(x) \exp\{-j2\pi x v\}\, dx \ .$$

Then the following are a Fourier transform pair is available from a standard table such as Campbell & Foster (1948).

$$
\begin{aligned}
f(x) &= \exp\left\{ -j\alpha x^2 \right\} \\
F(v) &= \sqrt{\frac{\pi}{\alpha}} \exp\left\{ j\left(\frac{(\pi v)^2}{\alpha} - \frac{\pi}{4} \right) \right\} = \sqrt{\frac{\pi}{j\alpha}} \exp\left\{ j\left(\frac{(\pi v)^2}{\alpha} \right) \right\}
\end{aligned}
$$

So then taking $\alpha = -\frac{\pi}{\lambda d}$, we have the new 1D Fourier transform pair given by

$$
\begin{aligned}
h_1(x) &= \frac{1}{\sqrt{j\lambda d}} \exp\left\{ j\pi \frac{x^2}{\lambda d} \right\} \\
H_1(v) &= \frac{1}{\sqrt{j\lambda d}} \sqrt{j\lambda d} \exp\left\{ -j\pi\lambda d v^2 \right\} \\
&= \exp\left\{ -j\pi\lambda d v^2 \right\} \ .
\end{aligned}
$$

Notice that the Fresnel kernel is a separable function given by (1) can be expressed as

$$h(x,y) = \exp\left\{j2\pi\frac{d}{\lambda}\right\} h_1(x)h_1(y)$$

So we have that

$$\begin{aligned} H(v_x, v_y) &= \exp\left\{j2\pi\frac{d}{\lambda}\right\} H_1(v_x)H_1(v_y) \\ &= \exp\left\{j2\pi\frac{d}{\lambda}\right\} \exp\left\{-j\pi\lambda d\left(v_x^2 + v_y^2\right)\right\} \end{aligned}$$