ECE 637 Final Exam

May 5, Spring 2021

Question 1

Rules: (2 points)I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

Upload a scan of your signature here:

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\mathrm{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}\left[X(f)\right]$$

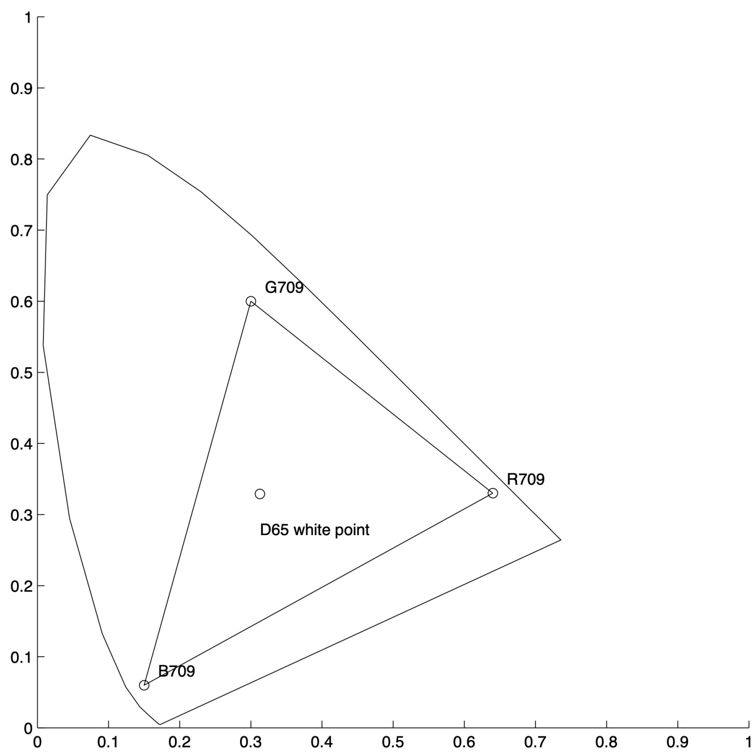
- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

Question 2 Colorimetry

(35 points)Consider the standard chromaticity diagram below, and for all questions assume that standard 709 r, g, b color primaries are used.

Instructions: For each of the sub-questions, you can either print the diagram, or you can redraw or sketch it. However, you should upload a separate diagram for each sub-question.



(1)(5 points) Draw the region on the diagram corresponding to real colors such that $r < 0$, $g > 0$, $b > 0$, and label this region "negative red".

(2)(5 points) Draw the region on the diagram corresponding to real colors such that $r < 0$, $g > 0$, $b < 0$, and label this region "negative red and blue".

(3)(5 points) Sketch to set of colors that can be generated by a combination of the red 709 primary and the D65 white point.

(4)(5 points) Draw points corresponding to the color primaries for X, Y, Z, and label the three points "X-primary", "Y-primary", and "Z-primary".

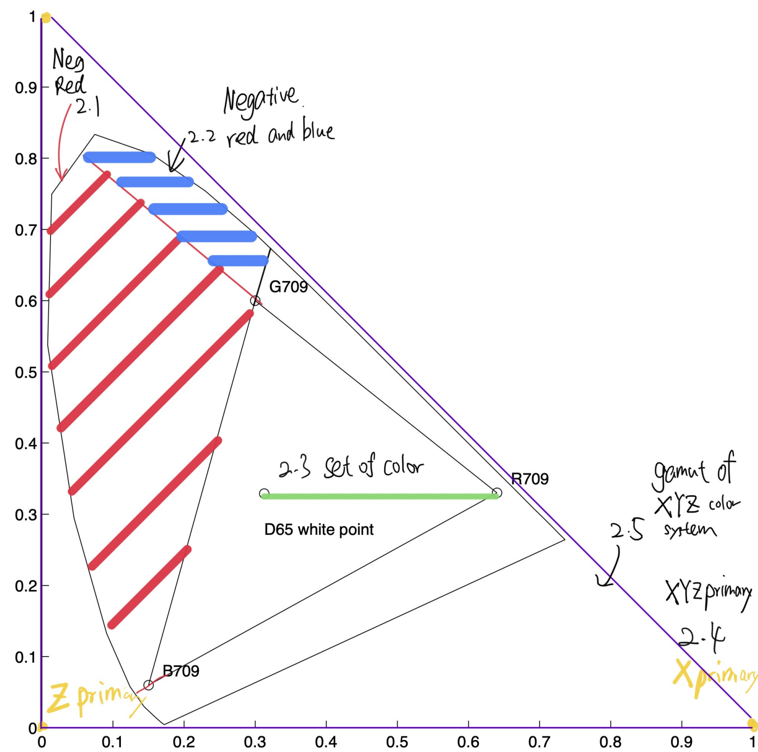
(5)(5 points) Draw a triangle corresponding to the gamut of the X,Y,Z color system, when the three tristimulus values are assumed positive.

(6)(5 points) Do all positive values of X, Y, Z correspond to real colors? Why or why not?

(7)(5 points) Do all real colors correspond to positive values of X,Y,Z? Why or why not?

Solution:

Q 2.1-Q 2.5



Q 2.6

No, not all positive values of X, Y, and Z correspond to real colors.

Only colors inside "horse shoe" are real colors. Colors outside that horse shoe region but within triangle formed by the X, Y, Z primaries are imaginary.

Q 2.7

Yes, all real color are contained in the "house shoe" and values in "house shoe" correspond to positive values of X, Y, Z.

Since the triangle inscribed by the X-primary, Y-primary, and Z-primary include the "horse shoe", it must be that all real colors correspond to positive values of the X, Y, Z tristimulus values.

Question 3 Nonlinear Estimation

(35 points)Consider a non-linear prediction problem for which you are trying to predict the value of a scalar X_n from a vector of observations Y_n. Our assumption is that we can estimate

$$X_n$$

using the non-linear predictor given by

$$\hat{X}_n = f_\theta(Y_n)$$

where $\theta \in \Re^p$ is a p dimensional parameter vector that controls the behavior of the nonlinear predictor.

We are given independent pairs of data with the form (Y_n, X_n). The data pairs are partitioned into two sets. The first set, $n \in S_1$, contains $N = |S_1|$ pairs, and the second set, $n \in S_2$, contains $M = |S_2|$ pairs. Using these data, we define the following quantities

$$MSE_1(\theta) = \frac{1}{N} \sum_{n \in S_1} \| X_n - f_\theta(Y_n) \|^2 \ ,$$

$$MSE_2(\theta) = \frac{1}{M} \sum_{n \in S_2} \| X_n - f_\theta(Y_n) \|^2 \ ,$$

$$MSE_3(\theta) = E\left[\| X_n - f_\theta(Y_n) \|^2 \right] \ .$$

Based on these error measures, we also define the following two estimates for the parameter vector.

$$\hat{\theta} = \arg\min_\theta MSE_1(\theta)$$

$$\theta^* = \arg\min_\theta MSE_3(\theta)$$

(1)(5 points) Which value would you expect to be smaller, $MSE_1(\hat{\theta})$ or $MSE_2(\hat{\theta})$. Why?

(2)(5 points) Which value would you expect to be smaller, $MSE_2(\hat{\theta})$ or $MSE_2(\theta^*)$. Why?

(3)(5 points) Which value would you expect to be smaller, $MSE_1(\hat{\theta})$ or $MSE_1(\theta^*)$. Why?

(4)(5 points) Sketch the plots of $MSE_1(\hat{\theta})$, $MSE_2(\hat{\theta})$, and $MSE_3(\theta^*)$ as a function of the amount of training data N.

(5)(5 points) Approximately how large should N be in order for $\hat{\theta}$ to be useful?

(6)(5 points) When writing a paper, should one report the value of $MSE_1(\hat{\theta})$ or $MSE_2(\hat{\theta})$? Why?

(7)(5 points) What names are conventionally given to the two sets, S_1 and S_2?

Solution:

Q 3.1

$MSE_1(\hat{\theta})$ is smaller. The parameter $\hat{\theta}$ is estimated to minimize the MSE of set S_1. However, it does not necessarily minimizes $MSE_2(\hat{\theta})$. In conventional terms, $MSE_1(\hat{\theta})$ is known as the training loss, which in general, is less than the testing loss of $MSE_2(\hat{\theta})$.

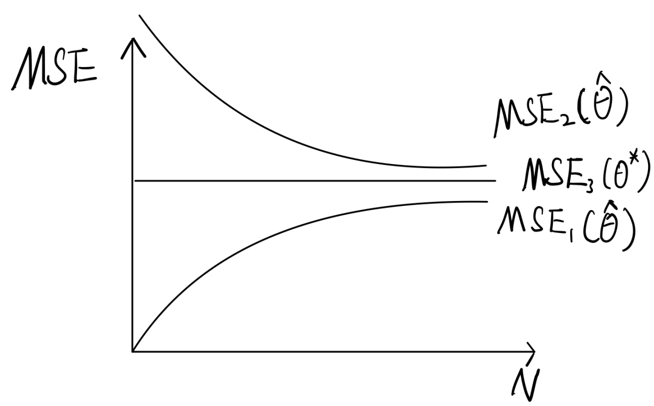
Q 3.2

$MSE_2(\theta^*)$ is smaller. θ^* is acquired by minimizing the expectation of the mean square error, which tends to work well when the samples in S_2 resembles the original distribution. The parameter $\hat{\theta}$ is estimated to minimize the MSE of set S_1. However, it does not necessarily work well for $MSE_2(\hat{\theta})$.

Q 3.3

$MSE_1(\hat{\theta})$ is smaller. Since $\hat{\theta}$ was determined by training on S_1 data, using it to calculate the error for S_1 should give a lower value than using θ^*, which minimizes the error for the entire distribution.

Q 3.4



Q 3.5

$N >= q$. Need at least N equations to solve p dimension θ.

Q 3.6

$MSE_2(\hat{\theta})$. When writing a paper, you should test on a different set of data than you train on!

Q 3.7

S_1 is training data set.

S_2 is testing data set.

Question 4 2D Sampling

(30 points) Consider a focal plane array with detectors of size $T \times T$. Let $g(x, y)$ denote the incoming light field, and let $s(m, n)$ denote the measurement from the $(m, n)^{th}$ detector. Then these are related by

$$s(m,n) = \int_{x=-T/2+mT}^{T/2+mT} \int_{y=-T/2+nT}^{T/2+nT} g(x,y)\, dx\, dy \ .$$

The sampled image is then displayed as

$$f(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(m,n) p(x - mT, y - nT) \ ,$$

where $p(x, y)$ is a 2D function.

(1)(5 points) If the image is viewed on an achromatic LCD display, then give a reasonable choice of the function $p(x, y)$ to accurately model the display.

(2)(5 points) Calculate an expression for, $S(e^{j\mu}, e^{j\nu})$, the DSFT of $s(m, n)$ in terms of the function $G(u, v)$ the CSFT of $g(x, y)$.

(3)(5 points) In order to meet the Nyquist rate for this sampling system, what constraints should $g(x, y)$ meet?

(4)(5 points) Assuming that the sampling system meets the Nyquist rate, then write a simplified expression for $S(e^{j\mu}, e^{j\nu})$ in terms of $G(u, v)$ when $|\mu| < \pi$ and $|\nu| < \pi$.

(5)(5 points) Assuming that the sampling system meets the Nyquist rate, find an equation that directly relates $G(u, v)$ and $F(u, v)$ for the LCD display.

(6)(5 points) Assuming that the sampling system meets the Nyquist rate, what discrete-space filter, $H(e^{j\mu}, e^{j\nu})$, should be applied to $s(m, n)$ in order to insure that $f(m, n) = g(m, n)$? Be specific.

Solution:

Q 4.1

For an LCD display, each pixel will be a square or rectangle that fills the region on the display. So we have that

$$p(x,y) = \text{rect}(x/T, y/T) \ .$$

Q 4.2

The effect of the sensor is to convolve the input signal, $g(x, y)$, with an PSF of

$$h(x,y) = \frac{1}{T^2} \text{rect}(x/T, y/T) \ .$$

The CSFT of $h(x, y)$ is given by

$$H\,(u,v) = \text{sinc}(Tu, Tv)$$

So therefore, the CSFT of the incoming signal after convolution with the PSF of $h(x, y)$ is given by

$$\tilde{G}(u, v) = \text{sinc}(Tu, Tv) \cdot G(u, v)$$

This results in the following sampled signal.

$$\begin{aligned} S\left(e^{j\mu}, e^{j\nu}\right) &= \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \tilde{G}\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right) \\ &= \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \text{sinc}\left(\frac{\mu - 2\pi k}{2\pi}, \frac{\nu - 2\pi l}{2\pi}\right) \cdot G\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right) \end{aligned}$$

Q 4.3

The system has sampling frequency, $\frac{1}{T}$. The max frequency of $g(x, y)$ should be less than $\frac{1}{2T}$. For $G(u, v)$, when $|\mu| \geq \frac{1}{2T}$ or $|\nu| \geq \frac{1}{2T}$, $G(u, v) = 0$

Q 4.4

If there is no aliasing, then for $|u| < \pi$ and $|v| < \pi$, we have that

$$S\left(e^{j\mu}, e^{j\nu}\right) = \frac{1}{T^2} \text{sinc}\left(\frac{\mu}{2\pi}, \frac{\nu}{2\pi}\right) \cdot G\left(\frac{\mu}{2\pi T}, \frac{\nu}{2\pi T}\right)$$

Q 4.5

The DSFT of the display psf is given by

$$P(u, v) = T^2 \, \text{sinc}(Tu, Tv) \ .$$

We you reconstruct a signal, you make the following substitutions.

$$\begin{aligned} \mu &\to 2\pi T u \\ \nu &\to 2\pi T v \end{aligned}$$

So by combining the reconstruction with the convolution with $p(x, y)$, then for $|u| < \frac{1}{2T}$ and $|v| < \frac{1}{2T}$, we have that

$$\begin{aligned} F(u, v) &= P(u, v) S\left(e^{j2\pi Tu}, e^{j2\pi Tv}\right) \\ &= T^2 \, \text{sinc}(Tu, Tv) \frac{1}{T^2} \, \text{sinc}\left(T\mu, T\nu\right) \cdot G\left(\mu, \nu\right) \end{aligned}$$

So that

$$F(u, v) = \text{sinc}^2(Tu, Tv) \cdot G\left(\mu, \nu\right) \ .$$

Q 4.6

In order to compensate for the roll off in frequency, we must compensate with

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{\text{sinc}^2(\frac{\mu}{2\pi}, \frac{\nu}{2\pi})}$$