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ECE 637 Midterm II

April 25, Spring 2021

Question 1

Rules: (2 points)I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

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Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\text{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\text{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}}[X(f)]$$

$$\text{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

Question 2 Random Processes

(40 points)Consider a sequence of i.i.d. random variables $X_n \sim N(0,1)$ for $n = \cdots, -1, 0, 1, \cdots$. System 1 denoted by $Y = T_1[X]$ has the input/output responses specified by

$$Y_n = \alpha Y_{n-1} + X_n \ ,$$

where $|\alpha| < 1$. System 2 denoted by $Z = T_2[Y]$ has the input/output responses specified by

$$Z_n = Y_n + Y_{n-1} \ .$$

(1)(10 points) a) Calculate the autocorrelation of X_n denoted by $R_x(k)$, and b) calculate the power spectrum of X_n denoted by $S_x(e^{j\omega})$.

(2)(10 points) For the system T_1, a) Calculate the impulse response, h_n, and b) calculate the frequency response $H(e^{j\omega})$.

(3)(10 points) For the system T_2, a) Calculate the impulse response, g_n, and b) calculate the frequency response $G(e^{j\omega})$ in simplified form.

(4)(5 points) Calculate the power spectrum for the signal Y_n denoted by $S_y(e^{j\omega})$.

(5)(5 points) Calculate the power spectrum for the signal Z_n denoted by $S_z(e^{j\omega})$.

Solution:

Q 2.1

(a)Since random variables $X_n \sim N(0,1)$ is i.i.d.,

$$R_{sr} = E[X_s X_r] = \begin{cases} 0, & s \neq r \\ E[X_s^2], & s = r \end{cases}$$

Since $X_n \sim N(0,1)$, $E[x_n] = 0$

$$\mathrm{Var}(x_n) = E\left[x_n^2\right] - E[x_n]^2 = 1$$

$$E\left[x_n^2\right] = 1$$

Therefore,

$$R_x(k) = \delta(k), k = s - r$$

(b) Given s,

$$\mu_s = E[X_s] = 0 = E[X_0]$$

Given s and r,

$$\begin{aligned} C_{sr} &= E[(x_s - \mu_s)(x_r - \mu_r)] \\ &= E[x_s x_r] \\ &= \begin{cases} 0 & , s \neq r \\ 1, & s = r \end{cases} \end{aligned}$$

Therefore, X_n is wide sense stationary.

$$\begin{aligned} S_x\left(e^{j\mu}, e^{j\nu}\right) &= \text{CSFT}\left[R_x(k)\right] \\ &= 1 \end{aligned}$$

Q 2.2

(a),(b) Calculate DTFT for system T_1,

$$\begin{aligned} Y(e^{j\omega}) &= \alpha e^{-j\omega} Y(e^{j\omega}) + X(e^{j\omega}) \\ \left(1 - \alpha e^{-j\omega}\right) Y(e^{j\omega}) &= X(e^{j\omega}) \\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} &= \frac{1}{1 - \alpha e^{-j\omega}} \end{aligned}$$

According to DTFT pairs,

$$h_n = \alpha^n u(n)$$

Q 2.3 We have,

$$Z_n = Y_n + Y_{n-1}$$

(a) $g_n = \delta(n) + \delta(n-1)$

(b) Calculate DTFT for system T_2,

$$\begin{aligned} Z(e^{j\omega}) &= Y(e^{j\omega}) + e^{-j\omega} Y(e^{j\omega}) \\ Z(e^{j\omega}) &= (1 + e^{-j\omega}) Y(e^{j\omega}) \\ G(e^{j\omega}) &= \frac{Z(e^{j\omega})}{Y(e^{j\omega})} \\ &= 1 + e^{-j\omega} \\ &= 2e^{-j\omega/2} \frac{e^{j\omega/2} + e^{-j\omega/2}}{2} \\ &= 2\cos(\omega/2)\, e^{-j\omega/2} \end{aligned}$$

Q 2.4

$$\begin{aligned} S_y\left(e^{j\omega}\right) &= S_x\left(e^{j\omega}\right) \cdot \left|H\left(e^{j\omega}\right)\right|^2 \\ &= \left|\frac{1}{1 - \alpha e^{-j\omega}}\right|^2 \\ &= \frac{1}{|1 - \alpha e^{-j\omega}|^2} \\ &= \frac{1}{|1 - \alpha\cos\omega + j\alpha\sin\omega|^2} \\ &= \frac{1}{(1 - \alpha\cos\omega)^2 + \alpha^2\sin^2\omega} \\ &= \frac{1}{1 - 2\alpha\cos\omega + \alpha^2\cos^2\omega + \alpha^2\sin^2\omega} \\ &= \frac{1}{1 + \alpha^2 - 2\alpha\cos\omega} \end{aligned}$$

Q 2.5

$$\begin{aligned}
S_z\left(e^{j\omega}\right) &= S_y\left(e^{j\omega}\right)\cdot\left|G\left(e^{j\omega}\right)\right|^2\\
&= \frac{1}{1+\alpha^2-2\alpha\cos\omega}\cdot|1+e^{-j\omega}|^2\\
&= \frac{|1+\cos\omega-j\sin\omega|^2}{1+\alpha^2-2\alpha\cos\omega}\\
&= \frac{1+2\cos\omega+\cos^2\omega+\sin^2\omega}{1+\alpha^2-2\alpha\cos\omega}\\
&= \frac{2(1+\cos\omega)}{1+\alpha^2-2\alpha\cos\omega}
\end{aligned}$$

Question 3 Eigen and Singular Value Decomposition

(20 points)Let

$$X = [x_1, \cdots, x_N]$$

where x_n are i.i.d. zero-mean jointly Gaussian random vectors of dimension p with distribution $N(0, R)$. Furthermore, define

$$\hat{R} = \frac{1}{N} X X^t \ ,$$

and let

$$X = U \Sigma V^t$$

be the singular value decomposition (SVD) of X, and let

$$\hat{R} = E \Lambda E^t$$

be the eigen value decomposition of $\hat{R}$, and define the distribution

$$p_{\hat{R}}(x) = \frac{1}{(2\pi)^{p/2}} |\hat{R}|^{-1/2} \exp\left\{ -\frac{1}{2} x^t \hat{R}^{-1} x \right\} \ .$$

(1)(5 points) Derive a simple expression for $E\left[\hat{R}\right]$.

(2)(5 points) Derive a simple expression for the eigenvectors, E, and eigenvalues, Λ, of $\hat{R}$ in terms of the SVD of X.

(3)(10 points) Assume that $p = 2$ and $U = [u_1, u_2]$, $V = [v_1, v_2]$, $\Sigma = \text{diag}\{\lambda_1, \lambda_2\}$. Then sketch and appropriately label the 2D ellipse defined by

$$p_{\hat{R}}(x) = \frac{1}{2\pi} \exp\left\{ -\frac{1}{2} \right\} \ .$$

Solution:

Q 3.1

$$\hat{R} = \frac{1}{N} X X^T = \frac{1}{N} \sum_{i=1}^{N} x_i x_i^T$$

$$\begin{aligned} E[\hat{R}] &= \frac{1}{N} E\left[\sum_{i=1}^{N} x_i x_i^T \right] \\ &= \frac{1}{N} \sum_{i=1}^{N} E\left[x_i x_i^T \right] \because X \sim N(0, R) \\ &= \frac{1}{N} \sum_{i=1}^{N} R \\ &= \frac{1}{N} \cdot N \cdot R \\ &= R \end{aligned}$$

Q 3.2

$$\hat{R} = \frac{1}{N} X X^T (\because X = U \Sigma V^T)$$
$$\hat{R} = \frac{1}{N} U \Sigma V^T \left(U \Sigma V^T \right)^T$$
$$= \frac{1}{N} U \Sigma V^T V \Sigma U^T (\because V \text{ is orthonormal}, \Sigma \text{ is diagonal})$$
$$= \frac{1}{N} U \Sigma^2 U^T = E \Lambda E^T$$

Therefore, eigenvectors, $E = U$, eigenvalues, $\Lambda = \frac{1}{N}\Sigma^2$

Q 3.3

$$x^T \hat{R}^{-1} x = 1$$
$$\hat{R}\hat{R}^{-1} = I = \frac{1}{N} U \Sigma^2 U^T \cdot N U \Sigma^{-2} U^T$$
$$N x^T U \Sigma^{-2} U^T x = 1$$
$$\left(U^T x \right)^T N \Sigma^{-2} \left(U^T x \right) = 1$$

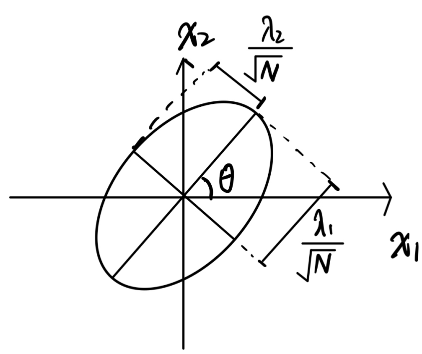
Let $y = U^T x$, then $y^T N \Sigma^{-2} y = 1$

$$\frac{N}{\lambda_1^2} y_1^2 + \frac{N}{\lambda_2^2} y_2^2 = 1$$
$$\frac{y_1^2}{\frac{\lambda_1^2}{N}} + \frac{y_2^2}{\frac{\lambda_2^2}{N}} = 1$$

Since $U \in \mathbb{R}^2$ is unitary, we can write

$$U^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = atan2(u_{21}, u_{11})$$



Question 4 Gamma Correction

(20 points) Consider a γ-corrected display with the discrete input $X(m,n) \in [0,255]$, and assume that the output intensity at the pixel (m,n) is given by

$$I(m,n) = I_o \left(\frac{X(m,n)}{255} \right)^{\gamma} ,$$

where $I(m,n)$ is the output intensity in units proportional to energy, and I_o is the maximum intensity output.

An emissive display is then used to display two different images. The first gray level image is given by $X_1(m,n) = g$. The second checker board image, $X_2(m,n)$, is given by

$$X_2(m,n) = \begin{cases} 255 & m+n \text{ is even} \\ 0 & m+n \text{ is odd} \end{cases} .$$

Imagine that you view the two images side-by-side from a large distance, so that you can no longer see the individual pixels in the checker-board image X_2, and you adjust the gray level to $g = g_0$ so that the two images, $X_1(m,n)$ and $X_2(m,n)$, match in intensity.

(1)(5 points) What average intensity (in units proportional to energy) do you see if you view the image corresponding to $X_1(m,n)$ from a large distance?

(2)(5 points) What average intensity (in units proportional to energy) do you see if you view the image corresponding to $X_2(m,n)$ from a large distance?

(3)(5 points) Calculate the value of g_0 that will cause $X_1(m,n)$ and $X_2(m,n)$ to match in average intensity.

(4)(5 points) Write an equation for the value of γ in terms of g_0.

Solution:

Q 4.1

$$I_1 = I_0 \left(\frac{g}{255} \right)^{\gamma}$$

Q 4.2

$$I_2 = \frac{1}{2} \left(I_0 \left(\frac{255}{255} \right)^{\gamma} + I_0 \left(\frac{0}{255} \right)^{\gamma} \right) = \frac{1}{2} I_0$$

Q 4.3

$$I_1 = I_2$$

$$I_0 \left(\frac{g_0}{255} \right)^{\gamma} = \frac{1}{2} I_0$$

$$g_0 = 255 \left(\frac{1}{2} \right)^{\frac{1}{\gamma}}$$

Q 4.4

$$g_0 = 255 \left(\frac{1}{2} \right)^{\frac{1}{\gamma}} \Rightarrow \gamma = \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{g_0}{255}\right)}$$

Question 5 Quantization

(20 points) Imagine that you have a gamma corrected image $X(m, n)$, and from this you compute an image that is linear in energy

$$I(m, n) = I_o \left(\frac{X(m, n)}{255} \right)^{\gamma} .$$

Then you gamma correct the image to form the displayed image

$$Y(m, n) = 255 \left(\frac{I(m, n)}{I_o} \right)^{1/\gamma} .$$

(1)(5 points) What data type should you use to store $I(m, n)$? Why?

(2)(5 points) What data type should you use to store $Y(m, n)$? Why?

(3)(10 points) What will happen if you store $I(m, n)$ using unsigned char for each pixel?

Solution:

Q 5.1 Floating point data. Because $I(m, n)$ is linear, you need to use floating point. The 8-bit Quantization of a linear model leads to big of steps in contrast in the dark regions, and it will cause contours in the dark region.

Q 5.2 Unsigned char or unsigned 8 bit integer. According to the given formula, $X(m, n)$ is in the range of $[0, 255]$. Also, after gamma correction, more detail is preserved during quantization. Therefore, Y can be stored with 8 bit precision in order to save storage space.

Q 5.3 It will cause contours in dark region.