PURDUE

ECE 63700

Exam #2, April 11, Spring 2025

NAME ____________________________________

PUID ____

Exam instructions:

- A fact sheet is included at the end of this exam for your use.
- You have 50 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

To ensure Gradescope can read your exam:

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

Name/PUID: Key

Problem 1. (25pt) White Noise Driven Random Process

Let X_n be a zero mean stationary Gaussian random process and assume that

$$\hat{X}_n = E\left[X_n | X_j \text{ for } j < n\right] = \sum_{k=1}^{p} X_{n-k} h_k \ ,$$

and then define the prediction errors, $W_n = X_n - \hat{X}_n$.

Problem 1a) What is the distribution of the prediction errors, W_n?

Solution:
Since the W_n are the prediction errors of a zero-mean Gaussian random process, they must be i.i.d. Gaussian random variables with some variance, σ_w^2. So we have that they are i.i.d. with $W_n \sim N(0, \sigma_w^2)$.

Problem 1b) Calculate the power spectrum, $S_w(e^{j\omega})$, of W_n.

Solution:
The autocorrelation of W_n is given by $R_w(k) = E\left[W_n W_{n+k}\right] = \sigma_w^2 \delta(k)$. So $S_w(e^{j\omega}) = DTFT\{R_w(n)\} = \sigma_w^2$.

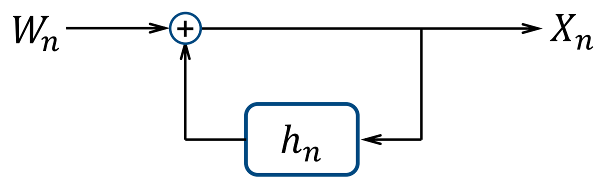
Problem 1c) Specify a filter using both a flow diagram and a recursive equation that generates X_n from the prediction errors, W_n.

Solution:
We know that

$$\begin{aligned} W_n &= X_n - \hat{X}_n \\ &= X_n - \sum_{k=1}^{p} X_{n-k} h_k \ . \end{aligned}$$

So we can rewrite that as the IIR recursion

$$X_n = W_n + \sum_{k=1}^{p} X_{n-k} h_k \ .$$



Problem 1d) Calculate the power spectrum, $S_x(e^{j\omega})$, of X_n.

Solution:
By taking the Fourier transform of the IIR recursion equation, we get

$$\begin{aligned} X(e^{j\omega}) &= W(e^{j\omega}) + \sum_{k=1}^{p} X(e^{j\omega}) e^{jk\omega} h_k \\ &= W(e^{j\omega}) + X(e^{j\omega}) \sum_{k=1}^{p} e^{jk\omega} h_k \\ &= W(e^{j\omega}) + X(e^{j\omega}) H(e^{j\omega}) \ . \end{aligned}$$

So we have that

$$X(e^{j\omega}) = \frac{1}{1 - H(e^{j\omega})} W(e^{j\omega}) \ .$$

So therefore, we have that

$$S_x(e^{j\omega}) = S_w(e^{j\omega}) \frac{1}{|1 - H(e^{j\omega})|^2}$$

$$S_x(e^{j\omega}) = \frac{\sigma_w^2}{|1 - H(e^{j\omega})|^2} \ .$$

Problem 1e) You are told that you need to generate a pseudo random process, $\tilde{X}_n$, on a computer that has the same distribution as X_n. Explain how you do it using both words and equations.

Solution:

First, you generate a sequence of pseudo random independent Gaussian random variables, $\tilde{W}_n$. Then you should filter the sequence, $\tilde{W}_n$, to produce the sequence, $\tilde{X}_n$, using the IIR filter

$$\tilde{X}_n = \tilde{W}_n + \sum_{k=1}^{p} \tilde{X}_{n-k} h_k \ .$$

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Problem 2. (25pt) Multivariate Gaussian

Let $X \sim N(0, R)$ be a Gaussian random $p-$dimensional vector, and let R have an eigen decomposition given by $R = E\Lambda E^t$, and define $B = R^{-1}$.

Problem 2a) Give a simplified expression for B in terms of E and a diagonal matrix, and show that, using this expression, $BR = I$.

Solution:

$$\begin{aligned} B &= R^{-1} \\ &= \left(E\Lambda E^t\right)^{-1} \\ &= (E^t)^{-1}\Lambda^{-1}E^{-1} \\ &= E\Lambda^{-1}E^t \end{aligned}$$

So then we have that

$$\begin{aligned} BR &= (E\Lambda^{-1}E^t)(E\Lambda E^t) \\ &= E\Lambda^{-1}E^tE\Lambda E^t \\ &= E\Lambda^{-1}\Lambda E^t \\ &= EE^t \\ &= I \ . \end{aligned}$$

Problem 2b) Let λ_k and e_k be the k^{th} eigenvalue and eigenvector of R.

- Give a precise specification of λ_k and e_k from R and Λ.
- Show that they solve the eigen value equation $Re_k = \lambda_k e_k$.

Solution:
e_k is the k^{th} column of E, and λ_k is the k^{th} diagonal element of the diagonal matrix Λ. So

then we have that

$$
\begin{aligned}
Re_k &= (E\Lambda E^t)e_k \\
&= E\Lambda \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots 0 \end{bmatrix} \\
&= \lambda_k E \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots 0 \end{bmatrix} \\
&= \lambda_k e_k
\end{aligned}
$$

Problem 2c) Define an transformation that generates a zero mean white random vector $W \sim N(0, I))$ from X.

Solution:

$$W = \Lambda^{-1/2} E^t X$$

Problem 2d) You are told that you need to generate a pseudo random vector, $\tilde{X}$, on a computer that has the same distribution as X. Explain how you do it using both words and equations.

Solution:
We can invert the whitening process to produce the equation

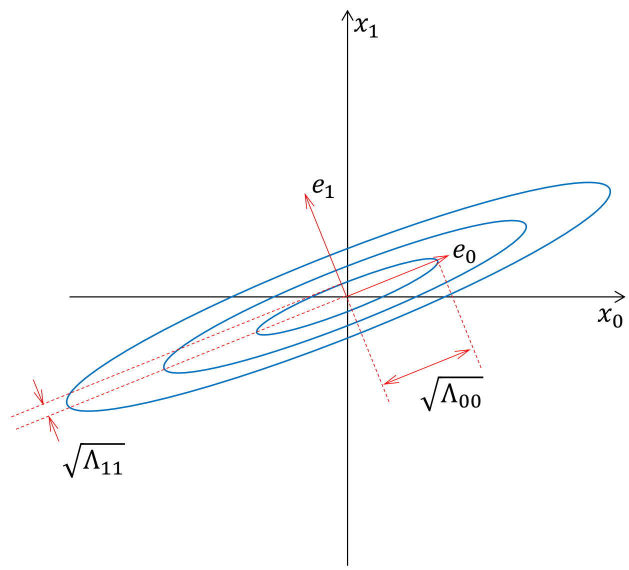
$$X = E\Lambda^{1/2} W \ .$$

So we first generate a pseudo random vector with components that are independent zero mean variance 1 Gaussian pseudo random numbers. So $\tilde{W} \in \mathbb{R}^p$ such that $\tilde{W}_i \sim N(0,1)$ for each component i. Then compute the following sequence of transformation to generate the desired pseudo random vector

$$\tilde{X} = E\Lambda^{1/2}\tilde{W} \ .$$

Problem 2e) Sketch the distribution of X in the 2D case (i.e., $p = 2$), and label the important characteristics of the sketch.

Solution:



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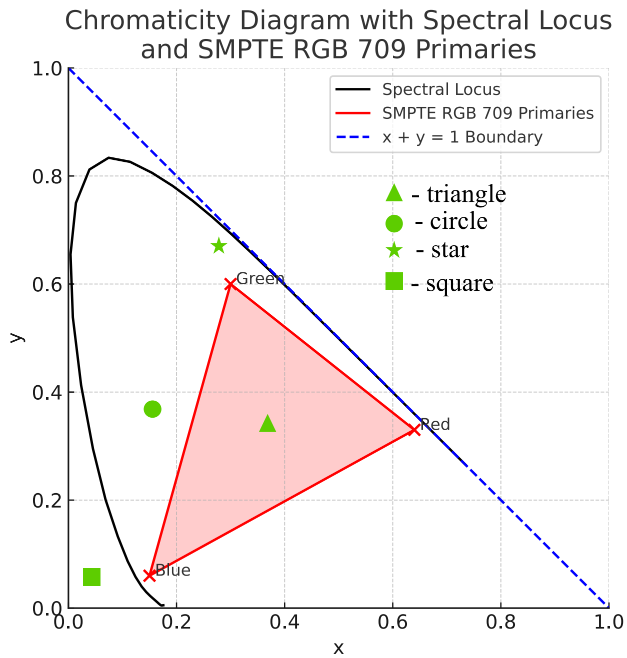
Problem 3. (20pt) Color Vision

In this problem, assume you are using an emissive display with SMPTE 709 RGB color primaries. Also, refer to the following chromaticity diagram.



Problem 3a) Let (r_t, g_t, b_t) denote the tristimuluous values indicated by the **triangle**.

- Is this a real color?
- Can this color be displayed?
- Are the values of r_t, g_t, and b_t less than or greater than zero?

Solution:
Yes, the color is real. Yes, the color can be displayed. $r_t > 0$, $g_t > 0$, and $b_t > 0$

Problem 3b) Let (r_c, g_c, b_c) denote the tristimuluous values indicated by the **circle**.

- Is this a real color?
- Can this color be displayed?

- Are the values of r_c, g_c, and b_c less than or greater than zero?

Solution:
Yes, the color is real. No, the color can not be displayed. $r_t < 0$, $g_t > 0$, and $b_t > 0$

Problem 3c) Let (r_s, g_s, b_s) denote the tristimuluous values indicated by the **star**.

- Is this a real color?
- Can this color be displayed?
- Are the values of r_s, g_s, and b_s less than or greater than zero?

Solution:
Yes, the color is real. No, the color can not be displayed. $r_t < 0$, $g_t > 0$, and $b_t < 0$

Problem 3d) Let (r_q, g_q, b_q) denote the tristimuluous values indicated by the **square**.

- Is this a real color?
- Can this color be displayed?
- Are the values of r_q, g_q, and b_q less than or greater than zero?

Solution:
No, the color is not real. No, the color can not be displayed. $r_t < 0$, $g_t > 0$, and $b_t > 0$

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Problem 4. (30pt) Sampling for Acquisition and Display

A CMOS sensor generates an output, $s(m,n)$, with the form

$$s(m,n) = \int_{\mathbb{R}^2} h(x - mT, y - nT)\ g(x,y) dx dy\ ,$$

where

$$h(x,y) = \frac{1}{T^2} \mathrm{rect}(x/T, y/T)\ ,$$

where $f_s = 1/T$.

The display then generates an output with the form

$$f(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} p(x - mT, y - nT)\ s(m,n)\ ,$$

where

$$p(x,y) = \mathrm{rect}(x/T, y/T)\ .$$

Problem 4a) Assuming that $g(x,y)$ is band limited to a frequency of f_c. Then what is the maximum value of f_c that allows for perfect reconstruction of $g(x,y)$ from $s(m,n)$.

Solution:

$$f_c = \frac{1}{2T} = \frac{1}{2} f_s$$

Problem 4b) Calculated an expression for $H(u,v)$, the CSFT of $h(x,y)$.

Solution:

$$H(u,v) = \mathrm{sinc}(Tu, Tv)$$

Problem 4c) Calculated an expression for $P(u,v)$, the CSFT of $p(x,y)$.

Solution:

$$P(u,v) = T^2 \mathrm{sinc}(Tu, Tv)$$

Problem 4d) Give an expression for $S(e^{j\mu}, e^{j\nu})$, the DSFT of $s(m,n)$, in terms of $H(u,v)$ and $G(u,v)$, the CSFT of $g(x,y)$.

Solution:

$$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} H\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi k}{2\pi T}\right) G\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi k}{2\pi T}\right)$$

Problem 4e) Give an expression for $F(u,v)$ the CSFT of $f(x,y)$ in terms of $P(u,v)$, $H(u,v)$ and $G(u,v)$, the CSFT of $g(x,y)$.

Solution:

$$\begin{aligned} F(u,v) &= P(u,v)\frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} H\left(\frac{2\pi uT - 2\pi k}{2\pi T}, \frac{2\pi vT - 2\pi l}{2\pi T}\right) G\left(\frac{2\pi uT - 2\pi k}{2\pi T}, \frac{2\pi vT - 2\pi l}{2\pi T}\right) \\ &= P(u,v)\frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} H\left(u - kf_s, u - lf_s\right) G\left(u - kf_s, u - lf_s\right) \end{aligned}$$

Problem 4f) What is the aggregate effect of this sampling and display system on the image. In other words, how does $f(x,y)$ differ from $g(x,y)$?

Solution:
The aggregate effect is to multiply the image by

$$\begin{aligned} F(u,v) &= P(u,v)H(u,v)G(u,v) \\ &= \frac{1}{T^2}\operatorname{sinc}^2(Tu,Tv)\ G(u,v) \end{aligned}$$

So this is a low pass filter.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X^*(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy \\ f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(f) &= Y\left(e^{j2\pi f T}\right) \end{aligned}$$