ECE 637 Midterm I

February 26, Spring 2021

Question 1

Rules: (2 points)I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

Upload a scan of your signature here:

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \end{aligned}$$

- CTFT Properties

$$x(-t) \stackrel{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \stackrel{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi ft_0}$$

$$x(at) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \stackrel{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_0 t} \stackrel{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \stackrel{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \stackrel{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)Y^*(f)df$$

- CTFT pairs

$$\mathrm{sinc}(t) \stackrel{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \stackrel{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \\ f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \stackrel{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

$$(n+1)a^n u(n) \stackrel{DTFT}{\Leftrightarrow} \frac{1}{(1-ae^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T[x(t)] = x(t)\sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T[x(t)] \stackrel{CTFT}{\Leftrightarrow} \frac{1}{T}\mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \stackrel{CTFT}{\Leftrightarrow} \frac{1}{T}\mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

Question 2

(36 points)Consider the following 2D system with input x(m, n) and output y(m, n),

$$y(m,n) = x(m,n) + \lambda \{x(m,n) - h(m,n) * x(m,n)\}$$

where $\lambda > 0$, and $h(m,n) = g(m)\, g(n)$ where

$$g(n) = \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+1) \ .$$

(1)(6 points) Calculate $H(e^{j\mu}, e^{j\nu})$, the DSFT of $h(m,n)$.

(2)(6 points) Calculate a closed form expression for $p(m,n)$, the point spread function of the overall system where $y(m,n) = p(m,n) * x(m,n)$.

(3)(6 points) Is $p(m,n)$ a separable function? Justify your answer.

(4)(6 points) Calculate a closed form expression for $P(e^{j\mu}, e^{j\nu})$, the DSFT of $p(m,n)$.

(5)(6 points) Sketch the 2D function $P(e^{j\mu}, e^{j\nu})$ for $(\mu, \nu) \in [-\pi, \pi]^2$.

(6)(6 points) What is the value of $P(e^{j\mu}, e^{j\nu})$ at $(\mu, \nu) = (0, 0)$? Explain the meaning of this answer.

Solution:

Q 2.1 First, we know

$$h(m,n) = g(m)\, g(n)$$

where,

$$g(n) = \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+1)$$

Then,

$$H\left(e^{j\mu}, e^{j\nu}\right) = G\left(e^{j\mu}\right) G\left(e^{j\nu}\right)$$

where,

$$\begin{aligned}
G\left(e^{j\omega}\right) &= \sum_n g(n) e^{j\omega n} \\
&= \sum_n \{\frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+1)\} e^{j\omega n} \\
&= \frac{1}{4}e^{j\omega} + \frac{1}{2}e^{j0} + \frac{1}{4}e^{-j\omega} \\
&= \frac{1}{2} + \frac{1}{4} * 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) \\
&= \frac{1}{2}(1 + \cos(\omega))
\end{aligned}$$

So therefore, we have that

$$H\left(e^{j\mu}, e^{j\nu}\right) = \frac{1}{4}(1 + \cos\mu)(1 + \cos\nu)$$

Q 2.2

$$p(m,n) = \delta(m,n) + \lambda\left\{\delta(m,n) - h(m,n) * \delta(m,n)\right\}$$

where $h(m,n) = g(m)\,g(n)$ and $g(n) = \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+1)$

Therefore,

$$\begin{aligned} p(m,n) = &-\frac{1}{16}\lambda\delta(m-1,n-1) - \frac{1}{8}\lambda\delta(m-1,n) - \frac{1}{16}\lambda\delta(m-1,n+1) \\ &-\frac{1}{8}\lambda\delta(m,n-1) + (1+\frac{3}{4}\lambda)\delta(m,n) - \frac{1}{8}\lambda\delta(m,n+1) \\ &-\frac{1}{16}\lambda\delta(m+1,n-1) - \frac{1}{8}\lambda\delta(m+1,n) - \frac{1}{16}\lambda\delta(m+1,n+1) \end{aligned}$$

Q 2.3 No, it is not separable.

Assume $p(m,n)$ a separable function, the filter point spread function can be decomposed as below,

$$\begin{vmatrix} -\frac{\lambda}{16} & -\frac{\lambda}{8} & -\frac{\lambda}{16} \\ -\frac{\lambda}{8} & 1+\frac{3\lambda}{4} & -\frac{\lambda}{8} \\ -\frac{\lambda}{16} & -\frac{\lambda}{8} & -\frac{\lambda}{16} \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \end{vmatrix} \times \begin{vmatrix} d & e & f \end{vmatrix} = \begin{vmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{vmatrix}$$

where $a, b, c, d, e, f \in R$

Compare below 2 relationships,

$$\frac{a}{b} = \frac{ad}{bd} = \frac{1}{2}$$

$$\frac{a}{b} = \frac{ae}{be} = \frac{-\lambda}{8+6\lambda}$$

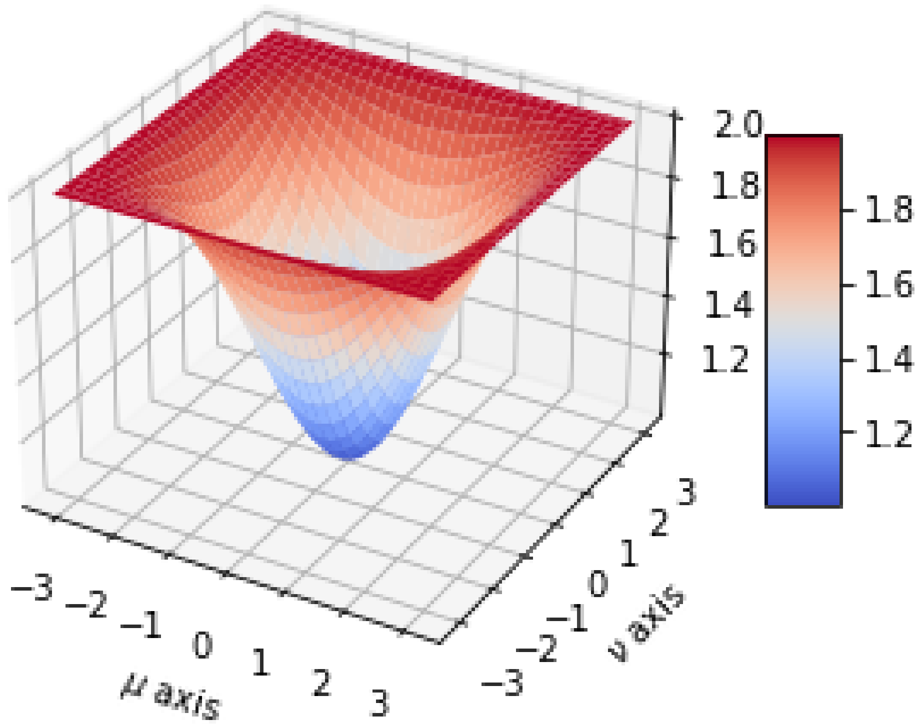
Notice that $\frac{1}{2} \neq \frac{-\lambda}{8+6\lambda}$ when $\lambda > 0$. This contradicts the assumption.

Therefore, $p(m,n)$ is not separable.

Q 2.4

$$\begin{aligned} P\left(e^{j\mu}, e^{j\nu}\right) &= (1+\lambda) - \lambda H\left(e^{j\mu}, e^{j\nu}\right) \\ &= (1+\lambda) - \frac{\lambda}{4}(1+\cos\mu)(1+\cos\nu) \end{aligned}$$

Q 2.5



Q 2.6

$$
\begin{aligned}
P\left(e^{j\mu}, e^{j\nu}\right) &= (1+\lambda) - \frac{\lambda}{4}(1+\cos\mu)(1+\cos\nu) \\
&= (1+\lambda) - \frac{\lambda}{4}(1+\cos 0)(1+\cos 0) \\
&= 1+\lambda-\lambda \qquad = 1
\end{aligned}
$$

The DC gain of the system is equal to 1.

Question 3

(34 points)Consider the 2D discrete space signal $x(m,n)$ with the DSFT of $X(e^{j\mu}, e^{j\nu})$ given by

$$X(e^{j\mu}, e^{j\nu}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m,n) e^{-j(m\mu + n\nu)} \ .$$

Then define the vertical and horizontal projecstions given by

$$p_v(n) = \sum_{m=-\infty}^{\infty} x(m,n)$$

$$p_h(m) = \sum_{n=-\infty}^{\infty} x(m,n)$$

with corresponding DTFTs given by

$$P_v(e^{j\omega}) = \sum_{n=-\infty}^{\infty} p_v(n) e^{-jn\omega}$$

$$P_h(e^{j\omega}) = \sum_{m=-\infty}^{\infty} p_h(m) e^{-jm\omega}$$

(1)(5 points) Derive an expression for $P_v(e^{j\omega})$ in terms of $X(e^{j\mu}, e^{j\nu})$.

(2)(5 points) Derive an expression for $P_h(e^{j\omega})$ in terms of $X(e^{j\mu}, e^{j\nu})$.

(3)(8 points) Derive an expression for $\sum_{n=-\infty}^{\infty} p_v(n)$ in terms of $X(e^{j\mu}, e^{j\nu})$.

(4)(8 points) Assume that you are able to measure the projections $p_v(n)$ and $p_h(m)$. Then what do these functions tell you about the function $X(e^{j\mu}, e^{j\nu})$? Sketch a figure that illustrates the answer to this problem.

(5)(8 points) Do the functions $p_v(n)$ and $p_h(m)$ together contain sufficient information to reconstruction the function $x(m,n)$? If so, provide a reconstruction algorithm; if not, provide a counter example.

Solution:

Q 3.1

$$\begin{aligned} P_v(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} p_v(n) e^{-jn\omega} \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m,n) e^{-jn\omega} e^{-jm0} \\ &= X(e^{j0}, e^{j\omega}) \end{aligned}$$

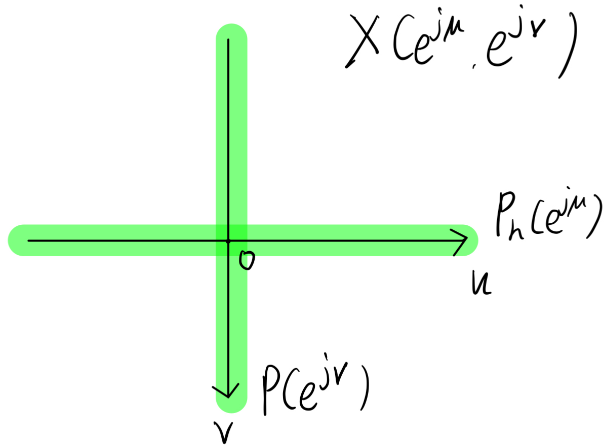
Q 3.2

$$
\begin{aligned}
P_h(e^{j\omega}) &= \sum_{m=-\infty}^{\infty} p_h(n) e^{-jm\omega} \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m,n) e^{-jm\omega} e^{-jn0} \\
&= X(e^{j\omega}, e^{j0})
\end{aligned}
$$

Q 3.3

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} p_v(n) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m,n) e^{-jn0} e^{-jm0} \\
&= X(e^{j0}, e^{j0})
\end{aligned}
$$

Q 3.4



These functions tell us two axes of the function $X(e^{j\mu}, e^{j\nu})$.

Q 3.5 The functions $p_v(n)$ and $p_h(m)$ together do not contain sufficient information to reconstruction the function $x(m,n)$. The counter example is below,

Assume we measured $p_v(n)$ and $p_h(m)$ as below,

n	1	2	3	otherwise
$p_v(n)$	12	15	18	0

m	1	2	3	otherwise
$p_h(m)$	6	15	24	0

We can find different $x(m,n)$ have same measurement. The counter example is below.

x(m,n) \ n / m	1	2	3	p$_h(m)$
1	1	2	3	6
2	4	5	6	15
3	7	8	9	24
p$_v(n)$	12	15	24	

x(m,n) \ n / m	1	2	3	p$_h(m)$
1	1	3	2	6
2	4	4	7	15
3	7	8	9	24
p$_v(n)$	12	15	24	

Question 4

(32 points) Let $g(x,y) = \texttt{sinc}(x/2, y/2)$and let $s(m,n) = g(mT, nT)$ for some $T > 0$.

(1)(8 points) Calculate $G(u,v)$ the CSFT of $g(x,y)$ and sketch it on $(u,v) \in [-2,2]^2$.

(2)(8 points) Calculate $S(e^{j\mu}, e^{j\nu})$ the DSFT of $s(m,n)$, and sketch it for $T = 1$ on $(\mu,\nu) \in [-2\pi, 2\pi]^2$.

(3)(8 points) What value of T corresponds to the Nyquist sampling rate?

(4)(8 points) Calculate the sampled signal, $s(m,n)$, and its Fourier transform, $S(e^{j\mu}, e^{j\nu})$, when $T = 1$.

Solution:

Q 4.1

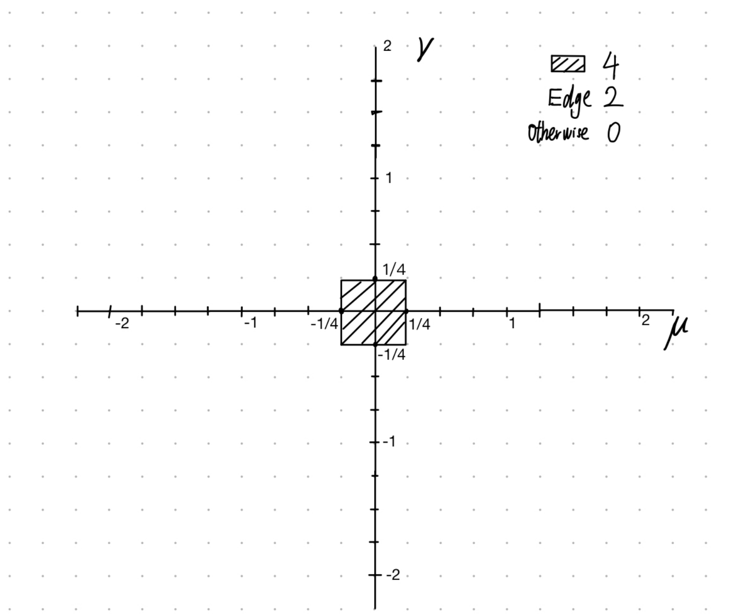
We have,

$$g(x,y) = \text{sinc}(x/2, y/2)$$

Since $\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$ and $z(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} Z(f/a)$, therefore,

$$G(u,v) = 4\,\text{rect}(2\mu, 2\nu)$$

Sketch it on $(u,v) \in [-2,2]^2$.



Q 4.2 According to Fact sheet,

$$y(n) = x(nT)$$
$$Y\left(e^{j\omega}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

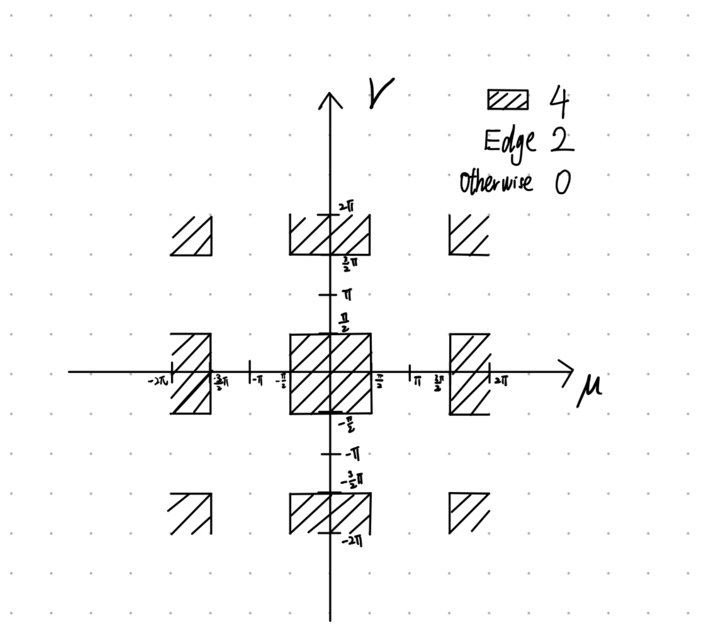
Now we have,

$$s(m,n) = g(mT, nT)$$

Therefore,

$$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi k}{2\pi T}\right)$$
$$= \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 4\,\mathrm{rect}\left(\frac{\mu - 2\pi k}{\pi T}, \frac{\nu - 2\pi k}{\pi T}\right)$$

Sketch it for $T = 1$ on $(\mu, \nu) \in [-2\pi, 2\pi]^2$.



Q 4.3

According to Q 4.1,

$$G(u,v) = 4\,\mathrm{rect}(2\mu, 2\nu)$$

the maximum frequency of $g(x,y)$ is $\frac{1}{4}$.

Therefore, to avoid aliasing, T should satisfy below condition,

$$f_s = \frac{1}{T} > 2 * \frac{1}{4} = \frac{1}{2}$$
$$0 < T < 2$$

Q 4.4 When $T = 2$,

$$s(m,n) = g(2m, 2n)$$
$$= \mathrm{sinc}(m,n)$$
$$= \frac{\sin(\pi m)}{\pi m} \times \frac{\sin(\pi n)}{\pi n}$$
$$= \delta(m,n)$$

where $m, n \in \mathcal{N}$

$$
\begin{aligned}
S(e^{j\mu}, e^{j\nu}) &= \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 4 \operatorname{rect}\left(\frac{\mu - 2\pi k}{\pi T}, \frac{\nu - 2\pi k}{\pi T}\right) \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \operatorname{rect}\left(\frac{\mu - 2\pi k}{2\pi}, \frac{\nu - 2\pi k}{2\pi}\right) \\
&= 1
\end{aligned}
$$