EE 641 Final Exam

Spring 1996

5/1/96

Instructions

The following is a take home exam.

- You are allowed 4 hours to complete the exam (9:30 AM to 1:30 PM). Hand in the exam to me at 1:30 PM **whether or not you have completed it.**
- Each problem is worth form 20 points to 30 points for a total of 100 points.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period. Your are allowed to use all class notes and handouts, and your EE600 text book. Your are not allowed to use supplementary information from the library, or publications not handed out in class.
- If you have any questions, call me at home 463-4378 (8:00AM to 8:00PM); or office 494-0340.

Good luck.

Problem 1.(20pt)

Let X be a strictly positive MRF on a 2-D lattice so that each point $s \in \{(s_1, s_2) : 1 \leq s_1 \leq N, 1 \leq s_2 \leq N\}$ has an associated pixel value $X_s \in \{0, 1\}$. Let A be the set of all cliques with the form

$$\begin{aligned}
&\{(s_1, s_2), (s_1 + 1, s_2)\} \\
&\{(s_1, s_2), (s_1 + 1, s_2 + 2)\} \\
&\{(s_1, s_2), (s_1 - 1, s_2), (s_1 - 1, s_2 - 1)\} \ .
\end{aligned}$$

Furthermore, assume that X has a Gibbs distribution with the form

$$p(x) = \frac{1}{z} \exp\left\{ \sum_{c \in A} V_c(x_c) \right\}$$

a) Find the smallest neighborhood system which is consistent with this set of cliques, A.

b) Find the complete set of cliques for the neighborhood of part a)

c) Express the function $p(x_{(N,N)} | x_{\partial(N,N)})$ in terms of the potential functions $V_c(x_c)$.

Problem 2.(30pt)

Let X, Y_1, Y_2, and Y_3 be binary random variables taking values in the set $\{0, 1\}$. Assume that Y_1, Y_2, and Y_3 are conditionally independent given X and that

$$p_{y_i|x}(y_i|x) = \begin{cases} 1-\alpha & \text{if } y_i = x \\ \alpha & \text{if } y_i \neq x \end{cases}$$

also assume that

$$p_x(x) = \begin{cases} \pi & \text{if } x = 0 \\ 1-\pi & \text{if } x = 1 \end{cases}$$

For each problem below, assume that you have performed an experiment yielding $Y_1 = 0$, $Y_2 = 0$, and $Y_3 = 1$.

a) Compute the conditional probability of X given Y_1, Y_2, and Y_3.

b) Compute the probability of Y_1, Y_2, and Y_3 as a function of π.

c) Use the this result of b) to compute the ML estimate of π.

d) Compute the EM update

$$\pi^{k+1} = \arg\max_{\pi} E[\log p_{x|y}(X|y, \pi)|Y = y, \pi^k] \ .$$

Problem 3.(25pt)

Let X, Y, Z be random fields defined on a 2-D lattice S. Further assume that

$$Y = h ** X + Z$$

where $**$ denotes 2-D convolution, $h_{(s_1,s_2)}$ is a strictly positive, low pass, blurring filter, and Z_s are i.i.d Gaussian random variables with mean 0 and variance σ^2. Also assume that X_s are i.i.d. Gaussian random variables with mean 0 and variance γ.

In the following answers, clearly state any approximations or assumptions.

a) Express the MAP estimate of X given Y in terms of a minimization of a matrix relationship.

b) Derive the gradient decent algorithm with step size μ for iteratively computing the MAP estimate, $\hat{X}$.

c) Derive an expression which gives the convergence rate of gradient decent as a function of 2-D frequency $\omega = (\omega_1, \omega_2)$.

d) Find an expression for the maximum stable value of μ.

Problem 4.(25pt)

From class we know that parameter estimation for MRF's is in general difficult. The following method, known as least squares parameter estimation, is a suboptimal, but tractable method of estimating parameters for discrete MRF's.

Let X be an Ising MRF with $X_s \in \{0, 1\}$ and

$$p(x) = \frac{1}{z} \exp \left\{ - \sum_{\{s,r\} \in C} \beta t(x_s, x_r) \right\}$$

where C is the set of nearest neighbor cliques, and $t(x_s, x_r) = 1$ if $x_s \neq x_r$ and $t(x_s, x_r) = 0$ otherwise. Furthermore, define K_s as the number of neighbors of X_s which are equal to 0. So $K_s \in \{0, 1, 2, 3, 4\}$.

a) Derive an expression for

$$f(x_{\partial s}) \triangleq \log \frac{p_{x_s|x_{\partial s}}(1|x_{\partial s})}{p_{x_s|x_{\partial s}}(0|x_{\partial s})}$$

in terms of K_s and β.

b) We may estimate the expression $f(x_{\partial s})$ by directly observing the rates at which the neighborhood configurations occur in a sample random field X. Derive an expression which counts these occurrances in X and estimates $f(x_{\partial s})$.

c) Equate the two expression of parts a) and b) and use this relation to compute an estimate for β.