ECE 641 Final Exam
Key

Q2.1

$$\forall i,j \quad P\{X_n = i, X_{n-1} = j\} = P\{X_n = j, X_{n-1} = i\}$$

$$\Updownarrow$$

$$\forall i,j \quad \pi_j P_{ji} = \pi_i P_{ij}$$

$\Uparrow$

These are the DBE's.

Q2.2

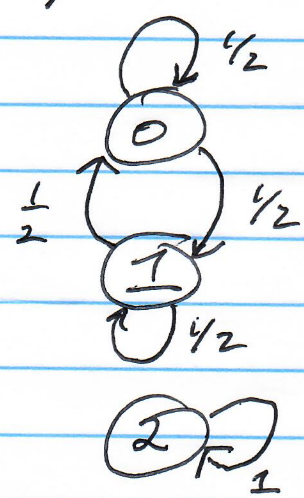
If $\quad \pi_j P_{ji} = \pi_i P_{ij} \quad \forall i,j$

$$\Rightarrow \sum_i \pi_j P_{ji} = \sum_i \pi_i P_{ij}$$

$$\pi_j = \sum_i \pi_i P_{ij}$$

Q 2.3

No, Take $M = 3$ with the following state transition diagram.



$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

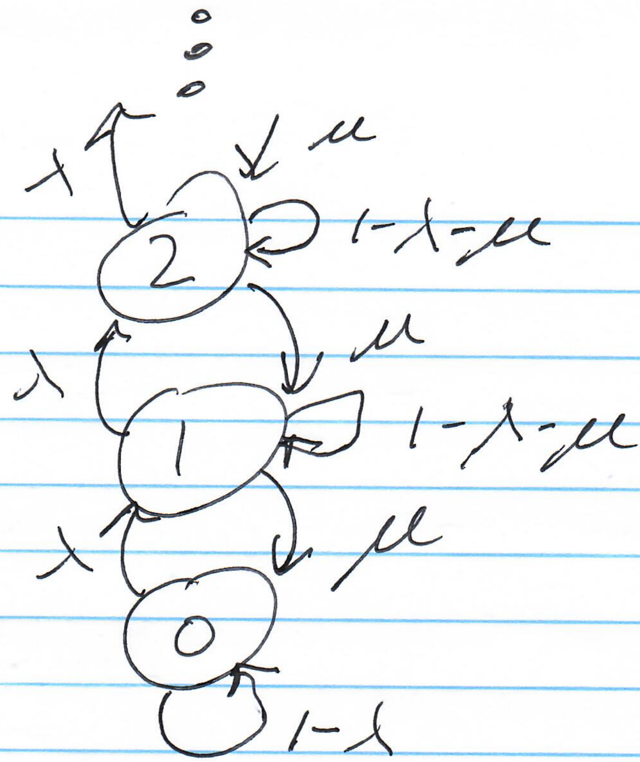
Then $\pi^{(0)} = [\,1/2 \;\; 1/2 \;\; 0\,]$

$\pi^{(2)} = [\,0 \;\; 0 \;\; 1\,]$

Both $\pi^{(0)}$ and $\pi^{(2)}$ meet the DBE, so both are stationary distributions, and X_n is reversible.

But $P^{\infty} = \begin{bmatrix} \pi^{(0)} \\ \pi^{(0)} \\ \pi^{(2)} \end{bmatrix}$

Q3.1



X_n is homogeneous because P_{ij} is not a function of time.

Q3.2

X_n is irreducible

Proof $\forall i, j$ without loose of generality, assume $i \geq j$

If $i = j + n$, then

$$[P^n]_{ji} > (\lambda)^n > 0$$

Also

$$[P^n]_{ij} > (\mu)^n > 0$$

$\implies$ irreducible

Q 3.3

Proof: take

$$\pi_i = \frac{1}{Z}\rho^i \qquad \rho = \left(\frac{\lambda}{\mu}\right)$$

then

$$Z = \sum_{i=0}^{M-1} \rho^i = \frac{1-\rho^M}{1-\rho}$$

$$\pi_i = \rho^i \left(\frac{1-\rho}{1-\rho^M}\right)$$

Also need to show for $j = i+1$

$$\frac{\lambda}{\mu} = \rho$$

$$\Downarrow$$

$$\rho^i \left(\frac{\lambda}{\mu}\right) = \rho^{i+1}$$

$$\Downarrow$$

$$\left(\frac{1-\rho}{1-\rho^M}\right)\rho^i\left(\frac{\lambda}{\mu}\right) = \left(\frac{1-\rho}{1-\rho^M}\right)\rho^{i+1}$$

$$\Downarrow$$

$$\pi_i \underset{\lambda}{\underset{\shortparallel}{P_{ij}}} = \pi_j \underset{\mu}{\underset{\shortparallel}{P_{ji}}}$$

$\Rightarrow$ DBE $\Rightarrow$ reversible

Q 3.4

X_n is not ergodic for

$$\rho = \frac{\lambda}{\mu} \geq 1 \text{ and } M = \infty$$

Intuitively, if $\left(\frac{\lambda}{\mu}\right) > 1$, then the arrival rate is greater or equal to the service rate, and the queue is unstable.

Q4.1

$$p(x_{s'} \mid x_r \; r\neq s') = \frac{p(x_{s'}, x_r \; r\neq s')}{\sum_{x_s} p(x_{s'}, x_r \; r\neq s')}$$

$$= \frac{\exp\left\{-\sum_{\{s,r\}\in P} b_{s,r}\,\rho(x_s - x_r)\right\}}{\int_{x_{s'}\in \mathbb{R}} \exp\left\{-\sum_{\{s,r\}\in P} b_{s,r}\,\rho(x_s - x_r)\right\} dx_{s'}}$$

since $b_{s,r} = b_{r,s}$

$$= \frac{\exp\left\{-\sum_{r\in\partial s'} b_{s',r}\,\rho(x_{s'} - x_r)\right\} \cancel{\exp\left\{-\sum_{\{s,r\}\in P'} b_{s,r}\,\rho(x_s - x_r)\right\}}}{\int_{x_{s'}\in \mathbb{R}} \exp\left\{-\sum_{r\in\partial s'} b_{s',r}\,\rho(x_{s'} - x_r)\right\} \cancel{\exp\left\{-\sum_{\{s,r\}\in P'} b_{s,r}\,\rho(x_s - x_r)\right\}} \, dx_{s'}}$$

where $P' = P - \{$any pair that contains $s'\}$

Q5.1

Gaussian Mixture distribution

Q 5.2

$$N_i = \sum_{n=1}^{N} \delta(X_n = i)$$

$$b_i = \sum_{n=1}^{N} Y_n \, \delta(X_n = i)$$

$$\hat{\pi}_i = \frac{N_i}{N}$$

$$\hat{\mu}_i = \frac{b_i}{N_i}$$

Q 5. 3

M-step

$$\hat{\pi}_i \leftarrow \frac{\bar{N}_i}{N}$$

$$\hat{\mu} \leftarrow \frac{\bar{b}_i}{\bar{N}_i}$$

where

E-step

$$\bar{N}_i = \sum_{n=1}^{N} P\{X_n = i \mid Y_n = y_n\}$$

$$\bar{b}_i = \sum_{n=1}^{N} y_n \, P\{X_n = i \mid Y_n = y_n\}$$

$$P\{X_n = i \mid Y_n = y_n\} = \frac{\pi_i \exp\{-\frac{1}{2\sigma^2}(y_n - \mu_i)^2\}}{\sum_k \pi_k \exp\{-\frac{1}{2\sigma^2}(y_n - \mu_k)^2\}}$$