$\boxed{\text{Lecture 2}}$

Y - Random Variable (upper case notation)

$$P(Y \le t) = F(t) = \int_{-\infty}^{t} p(\tau)\, d\tau$$

Note: $p(\tau)$ may not exist

RV or Number?

example 1)

Let $g(\circ)$ be a function

$$g(Y) \rightarrow RV$$

example 2)

$$E[Y] = \int_{-\infty}^{\infty} y\, p(y)\, dy \rightarrow \text{number}$$

example 3)

$$p(Y) \rightarrow RV \quad (\text{same as } 1)$$

example 4) X, Y both RV

$$E[Y \mid X = 3] \longrightarrow \text{number}$$

$$E[Y \mid X = x] = \int_{-\infty}^{\infty} y\, p_{y|x}(y \mid x)\, dy$$

$$\triangleq g(x) \text{ a function of } x$$

not a RV

example 5)

$$E[Y/X] = \int_{-\infty}^{\infty} y\, p_{y|x}(y/X)\, dy$$

$$= g(X)$$

function of a RV $\Rightarrow$ RV

example 6)

$$E[E[Y|X]] = E[g(X)]$$

$$= \int_{-\infty}^{\infty} g(x)\, p(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, p_{y|x}(y/x)\, dy\; p(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, p_{y,x}(y,x)\, dy\, dx$$

$$= E[Y]$$

$$E[Y] = E[E[Y/X]]$$

Estimation

$$P_\theta(Y \le t) = \int_{-\infty}^{t} p_\theta(y)\,dy$$

unknown parameter

- Use Y to guess at θ

$$\hat{\theta} = T(Y)$$

$T(\cdot)$ - estimator (function)
$\hat{\theta}$ - estimate (RV)
θ - parameter (number)

Maximum Likelihood (ML) Estimation

$$\underbrace{\hat{\theta}}_{RV} = \arg\max_{\theta} \underbrace{p_\theta(Y)}_{RV}$$

$$= \arg\max_{\theta} \log p_\theta(Y)$$

$$\left.\frac{d \log p_\theta(Y)}{d\theta}\right|_{\theta=\hat{\theta}} = 0$$

Example

$Y = (Y_1, Y_2, Y_3)$ iid $N(\mu, 1)$

$$p_\mu(y) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(y_i - \mu)^2\right\}$$

$$\log p_\mu(y) = \sum_{i=1}^{3} \left\{-\frac{1}{2}(y_i - \mu)^2 - \frac{1}{2}\log(2\pi)\right\}$$

$$\hat{\mu} = \underset{\mu}{\operatorname{argmax}} \log p_\mu(Y)$$

$$= \underset{\mu}{\operatorname{argmax}} \sum_{i=1}^{3} \left\{-\frac{1}{2}(Y_i - \mu)^2 - \cancel{\frac{1}{2}\log(2\pi)}\right\}$$

$$\frac{d}{d\mu} \log p_\mu(Y)\Big|_{\mu = \hat{\mu}}$$

$$= \sum_{i=1}^{3} (Y_i - \hat{\mu}) = 0$$

$$3\hat{\mu} = \sum_{i=1}^{3} Y_i$$

$$\hat{\mu} = \frac{Y_1 + Y_2 + Y_3}{3}$$

Bayesian Estimators

X - unknown RV

Y - observed RV

$\hat{X} = T(Y)$

$\hat{X}$ - Best guess at X

What does "best" mean?

(RV) $\longrightarrow C(X, \hat{X})$ - cost of guessing $\hat{X}$ when the correct answer is X

examples $\quad C(X, \hat{X}) = (X - \hat{X})^2$

$\qquad\qquad\qquad\qquad\quad |X - \hat{X}|$

Find $\hat{X}$ that minimizes

$$E[C(X, \hat{X})] = \text{average cost}$$

If $C(X, \hat{X}) = (X - \hat{X})^2$

Find $\hat{X} = T(Y)$ so that

$E[(X - \hat{X})^2]$ is minimum

minimum mean squared error estimate (MMSE)

What is $\hat{X} = T(Y)$?

$$\hat{X} = E[X / Y]$$

<u>example</u>

X – unobserved RV
Y – observed

$X \sim N(0, \sigma^2)$

$Y = X + \underset{\text{noise}}{N} \qquad N \sim N(0, \sigma_n^2)$

Example: $C(x,\hat{x}) = |x-\hat{x}|^2$

$\Rightarrow$ minimum mean squared error estimation

Solution: $\hat{X} = E[X|Y] = \mathcal{F}(Y)$

$$Y = X + N \qquad X \sim N(0,\sigma^2)$$
$$N \sim N(0,\sigma_n^2)$$

$$\hat{X} = \int_{-\infty}^{\infty} x \, p_{x|y}(x|y)\, dx$$

$$p_{x|y}(x|y) = \frac{p_{y|x}(y|x)\, p_x(x)}{p_y(y)}$$

$$p_{y|x}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_n^2} \exp\left\{ \frac{-1}{2\sigma_n^2}(y-x)^2 \right\}$$

$$p_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma^2} \exp\left\{ \frac{-1}{2\sigma_n^2} x^2 \right\}$$

$$p_{x|y}(x|y) = \frac{\frac{1}{2\pi\,\sigma_n^2\sigma^2} \exp\left\{ \frac{-1}{2\sigma_n^2}(y-x)^2 - \frac{1}{2\sigma^2}x^2 \right\}}{p_y(y)}$$

$$= \frac{1}{Z(y)} \exp\left\{ a(x-b)^2 + c \right\}$$

where $z(y) \leftarrow$ normalizing constant

$a, b, c \leftarrow$ constants

(1) $$\frac{-2}{2\sigma_n^2}(x-y) - \frac{2}{2\sigma^2}x$$

$$= 2a(x-b)$$

(2) $$\frac{-1}{\sigma_n^2} \frac{-1}{\sigma^2} = 2a$$

$$a = -\frac{1}{2}\left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma^2}\right)$$

from (1)

$$\left(\frac{-1}{\sigma_n^2} - \frac{1}{\sigma^2}\right)x + \frac{1}{\sigma_n^2}y = 2ax - 2ab$$

$$\frac{1}{\sigma_n^2}y = \left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma^2}\right)b$$

$$b = \frac{1/\sigma_n^2}{1/\sigma_n^2 + 1/\sigma^2}y$$

$$= \frac{\sigma^2}{\sigma^2 + \sigma_n^2}y$$

$$a = -\frac{1}{2}\left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma^2}\right)$$

$$b = \frac{\sigma^2}{\sigma^2 + \sigma_n^2}\, Y$$

$$\text{solution}\begin{cases} p_{x|y}(x|y) = \dfrac{1}{Z'(y)} \exp\left\{ -\dfrac{1}{2\sigma_{x|y}} (x - \mu_{x|y}) \right\} \\[2ex] \sigma_{x|y} = \dfrac{\sigma_n^2 \sigma^2}{\sigma_n^2 + \sigma^2} \\[2ex] \mu_{x|y} = \dfrac{\sigma^2}{\sigma^2 + \sigma_n^2}\, Y \end{cases}$$