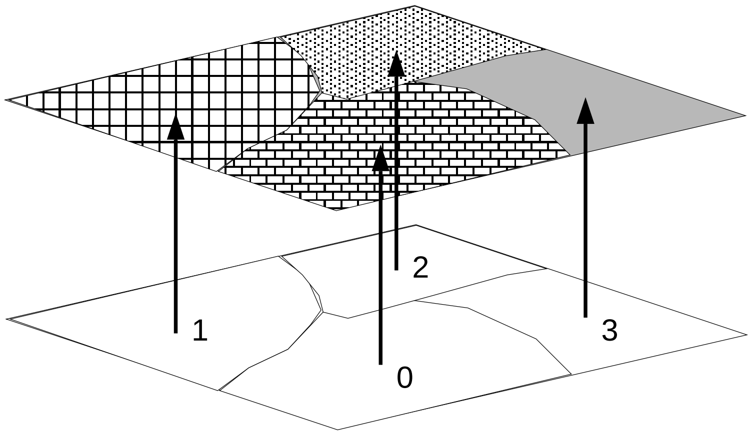
Application of MRF's to Segmentation

- Topics to be covered:
  - The Model
  - Bayesian Estimation
  - MAP Optimization
  - Parameter Estimation
  - Other Approaches

Bayesian Segmentation Model



- Discrete MRF is used to model the segmentation field.
- Each class is represented by a value $X_s \in \{0, \cdots, M-1\}$
- The joint probability of the data and segmentation is

$$P\{Y \in dy, X = x\} = p(y|x)p(x)$$

where

- $p(y|x)$ is the data model
- $p(x)$ is the segmentation model

Bayes Estimation

- $C(x, X)$ is the cost of guessing x when X is the correct answer.
- $\hat{X}$ is the estimated value of X.
- $E[C(\hat{X}, X)]$ is the expected cost (risk).
- Objective: Choose the estimator $\hat{X}$ which minimizes $E[C(\hat{X}, X)]$.

Maximum *A Posteriori* (MAP) Estimation

- Let $C(x, X) = \delta(x \neq X)$
- Then the optimum estimator is given by

$$\begin{aligned}
\hat{X}_{MAP} &= \arg\max_x p_{x|y}(x|Y) \\
&= \arg\max_x \log \frac{p_{y,x}(Y,x)}{p_y(Y)} \\
&= \arg\max_x \{\log p(Y|x) + \log p(x)\}
\end{aligned}$$

- Advantage:
  - Can be computed through direct optimization
- Disadvantage:
  - Cost function is unreasonable for many applications

Maximizer of the Posterior Marginals (MPM) Estimation[12]

- Let $C(x, X) = \sum_{s \in S} \delta(x_s \neq X_s)$
- Then the optimum estimator is given by

$$\hat{X}_{MPM} = \arg\max_{x_s} p_{x_s|Y}(x_s|Y)$$

- Compute the most likely class for each pixel
- Method:
  - Use simulation method to generate samples from $p_{x|y}(x|y)$.
  - For each pixel, choose the most frequent class.
- Advantage:
  - Minimizes number of misclassified pixels
- Disadvantage:
  - Difficult to compute

Simple Data Model for Segmentation

- Assume:
  - $x_s \in \{0, \cdots, M-1\}$ is the class of pixel s.
  - Y_s are independent Gaussian random variables with mean μ_{x_s} and variance $\sigma^2_{x_s}$.

$$p_{y|x}(y|x) = \prod_{s \in S} \frac{1}{\sqrt{2\pi\sigma^2_{x_s}}} \exp\left\{ -\frac{1}{2\sigma^2_{x_s}} \left(y_s - \mu_{x_s}\right)^2 \right\}$$

- Then the negative log likelihood has the form

$$-\log p_{y|x}(y|x) = \sum_{s \in S} l(y_s|x_s)$$

where

$$l(y_s|x_s) = -\frac{1}{2\sigma^2_{x_s}} \left(y_s - \mu_{x_s}\right)^2 - \frac{1}{2} \log\left(2\pi\sigma^2_{x_s}\right)$$

More General Data Model for Segmentation

- Assume:
  - Y_s are conditionally independent given the class labels X_s
  - $X_s \in \{0, \cdots, M-1\}$ is the class of pixel s.
- Then

$$-\log p_{y|x}(y|x) = \sum_{s \in S} l(y_s|x_s)$$

where

$$l(y_s|x_s) = -\log p_{y_s|x_s}(y_s|x_s)$$

MAP Segmentation

- Assume a prior model for $X \in \{0, \cdots, M-1\}^{|S|}$ with the form

$$\begin{aligned} p_x(x) &= \frac{1}{Z} \exp\{-\beta \sum_{\{i,j\} \in \mathcal{C}} \delta(x_i \neq x_j)\} \\ &= \frac{1}{Z} \exp\{-\beta t_1(x)\} \end{aligned}$$

  where $\mathcal{C}$ is the set of 4-point neighboring pairs

- Then the MAP estimate has the form

$$\begin{aligned} \hat{x} &= \arg\min_x \left\{ -\log p_{y|x}(y|x) + \beta t_1(x) \right\} \\ &= \arg\min_x \left\{ \sum_{s \in S} l(y_s|x_s) + \beta \sum_{\{i,j\} \in \mathcal{C}} \delta(x_i \neq x_j) \right\} \end{aligned}$$

- This optimization problem is very difficult

An Exact Solution to MAP Segmentation

- When $M = 2$, the MAP estimate can be solved exactly in polynomial time
  - See [9] for details.
  - Based on *minimum cut* problem and Ford-Fulkerson algorithm [5].
  - Works for general neighborhood dependencies
  - Only applies to binary segmentation case

Approximate Solutions to MAP Segmentation

- Iterated Conditional Models (ICM) [2]
  - A form of iterative coordinate descent
  - Converges to a local minima of posterior probability
- Simulated Annealing [6]
  - Based on simmulation method but with decreasing temperature
  - Capable of "climbing" out of local minima
  - Very computationally expensive
- MPM Segmentation [12]
  - Use simulation to compute approximate MPM estimate
  - Computationally expensive
- Multiscale Segmentation [3]
  - Search space of segmentations using a course-to-fine strategy
  - Fast and robust to local minima
- Other approaches
  - Dynamic programming does not work in 2-D, but approximate recursive solutions to MAP estimation exist[4, 13]
  - Mean field theory as approximation to MPM estimate[14]

Iterated Conditional Modes (ICM) [2]

- Minimize cost function with respect to the pixel x_r

$$
\begin{aligned}
\hat{x}_r &= \arg\min_{x_r}\left\{\sum_{s\in S} l(y_s|x_s) + \beta \sum_{\{i,j\}\in\mathcal{C}} \delta(x_i \neq x_j)\right\} \\
&= \arg\min_{x_r}\left\{l(y_r|x_r) + \beta \sum_{s\in\partial r} \delta(x_s \neq x_r)\right\} \\
&= \arg\min_{x_r}\left\{l(y_r|x_r) + \beta v_1(x_r, x_{\partial r})\right\}
\end{aligned}
$$

- Initialize with the ML estimate of X

$$
[\hat{x}_{ML}]_s = \arg\min_{0\leq m<M} l(y_s|m)
$$

ICM Algorithm

ICM Algorithm:

1. Initialize with ML estimate

$$x_s \leftarrow \arg \min_{0 \leq m < M} l(y_s|m)$$

2. Repeat until no changes occur

   (a) For each $s \in S$

$$x_s \leftarrow \arg \min_{0 \leq m < M} \{l(y_s|m) + \beta v_1(m, x_{\partial s})\}$$

- For each pixel replacement, cost decreases $\Rightarrow$ cost functional converges
- Variation: Only change pixel value when cost *strictly* decreases
- ICM + Variation $\Rightarrow$ sequence of updates converge in finite time
- Problem: ICM is easily trapped in local minima of the cost functional

Low Tempurature Limit for Gibb Distribution

- Consider the Gibbs distribution for the discrete random field X with tempurature parameter T

$$p_T(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{T} U(x) \right\}$$

- For $x \neq \hat{x}_{MAP}$, then $U(\hat{x}_{MAP}) < U(x)$ and

$$\begin{aligned} \lim_{T \downarrow 0} \frac{p_T(\hat{x}_{MAP})}{p_T(x)} &= \lim_{T \downarrow 0} \exp\left\{ \frac{1}{T} \left(U(x) - U(\hat{x}_{MAP}) \right) \right\} \\ &= \infty \end{aligned}$$

Since $p_T(\hat{x}_{MAP}) \leq 1$, we then know that $x \neq \hat{x}_{MAP}$

$$\lim_{T \downarrow 0} p_T(x) = 0$$

So if $\hat{x}_{MAP}$ is unique, then

$$\lim_{T \downarrow 0} p_T(\hat{x}_{MAP}) = 1$$

Low Temperature Simulation

- Select "small" value of T
- Use simulation method to generate sample X^* form the distribution

$$p_T(x) = \frac{1}{Z} \exp\left\{-\frac{1}{T}U(x)\right\}$$

- Then $p_T(X^*) \cong p_T(\hat{x}_{MAP})$
- Problem:

  T too large $\Rightarrow$ X^* is far from MAP estimate

  T too small $\Rightarrow$ convergence of simulation is **very** slow

- Solution:

  Let T go to zero slowly

  Known as simulated annealing

Simulated Anealing with Gibbs Sampler[6]

Gibbs Sampler Algorithm:

1. Set $N = \#$ of pixels
2. Select "annealing schedule": Decreasing sequence T_k
3. Order the N pixels as $N = s(0), \cdots, s(N-1)$
4. Repeat for $k = 0$ to ∞
   (a) Form $X^{(k+1)}$ from $X^{(k)}$ via

$$X_r^{(k+1)} = \begin{cases} W & \text{if } r = s(k) \\ X_r^{(k)} & \text{if } r \neq s(k) \end{cases}$$

where $W \sim p_{T_k}\left(x_{s(k)} \middle| X^{(k)}_{\partial s(k)}\right)$

- For example problem:

$$U(x) = \sum_{s\in S} l(y_s|x_s) + \beta t_1(x)$$

and

$$p_{T_k}(x_s|x_{\partial s}) = \frac{1}{z'} \exp\left\{-\frac{1}{T_k}\left(l(y_s|x_s) + \beta v_1(x_s, x_{\partial s}) \right)\right\}$$

Convergence of Simulated Annealing [6]

- Definitions:
  - N - number of pixels
  - $\Delta = \arg\max_x U(x) - \arg\min_x U(x)$
- Let

$$T_k = \frac{N\Delta}{\log(k+1)}$$

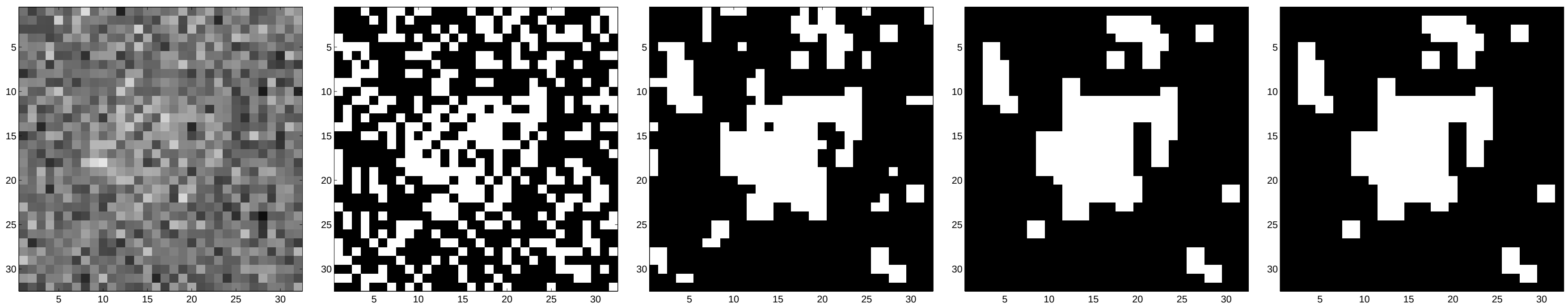
Theorem: The the simulation converges to $\hat{x}_{MAP}$ almost surely. [6]

- Problem: This is very slow!!!
- Example: $N = 10000$, $\Delta = 1 \Rightarrow T_{e^{10000}-1} = 1/2$.
- More typical annealing schedule that achieves approximate solution

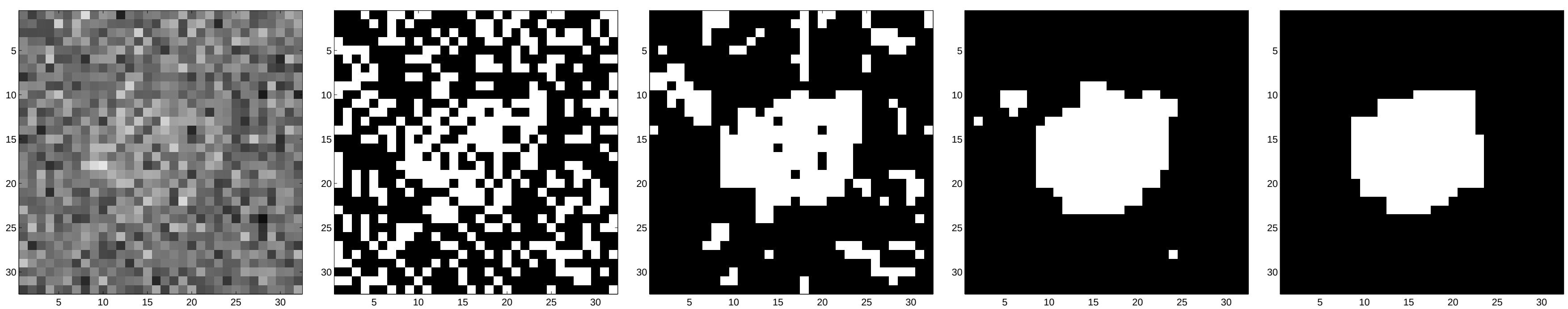
$$T_k = T_0 \left(\frac{T_K}{T_0} \right)^{k/K}$$

Segmentation Example

- Iterated Conditional Modes (ICM): ML ; ICM 1; ICM 5; ICM 10



- Simulated Annealing (SA): ML ; SA 1; SA 5; SA 10



Maximizer of the Posterior Marginals (MPM) Estimation[12]

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MPM Segmentation Algorithm [12]

- Define the function

$$X \leftarrow Simulate(X_{init}, p_{x|y}(x|y))$$

This function applies one full pass of a simulation algorithm with stationary distribution $p_{x|y}(x|y)$ and starting with initial value X_{init}.

MPM Algorithm:

1. Select parameters M_1 and M_2
2. For $i = 0$ to $M_1 - 1$
   (a) Repeat M_2 times

   $$X \leftarrow Simulate(X, p_{x|y}(x|y))$$

   (b) Set $X^{(i)} \leftarrow X$
3. For each $s \in S$, compute

$$\hat{x}_s \leftarrow \arg \max_{0 \leq m < M} \sum_{i=0}^{M_1 - 1} \delta\left(X^{(i)} = m\right)$$

Multiscale MAP Segmentation

- Renormalization theory[8]
  - Theoretically results in the exact MAP segmentation
  - Requires the computation of intractable functions
  - Can be implemented with approximation
- Multiscale segmentation[3]
  - Performs ICM segmentation in a coarse-to-fine sequence
  - Each MAP optimization is initialized with the solution from the previous coarser resolution
  - Used the fact that a discrete MRF constrained to be block constant is still a MRF.
- Multiscale Markov random fields[10]
  - Extended MRF to the third dimension of scale
  - Formulated a parallel computational approach

Multiscale Segmentation [3]

- Solve the optimization problem

$$\hat{x}_{MAP} = \arg\min_x \left\{ \sum_{s \in S} l(y_s|x_s) + \beta_1 t_1(x) + \beta_2 t_2(x) \right\}$$

- Break x into large blocks of pixels that can be changed simultaneously
- Make large scale moves can lead to
  - Faster convergence
  - Less tendency to be trapped in local minima

Formulation of Multiscale Segmentation [3]

- Pixel blocks
  - The s^{th} block of pixels

$$d^{(k)}(s) = \{(i,j) \in S : (\lfloor i/2^k \rfloor, \lfloor j/2^k \rfloor) = s\}$$

  - Example: If $k = 3$ and $s = (0,0)$, then $d^{(k)}(s) = [(0,0), \cdots, (7,0), (0,1), \cdots, (7,1), \cdots, (0,7), \cdots, (7,7)]$
- Coarse scale statistics:
  - We say that x is 2^k-*block-constant* if there exists an $x^{(k)}$ such that for all $r \in d^{(k)}(s)$

$$x_r = x_s^{(k)}$$

  - Coarse scale likelihood functions

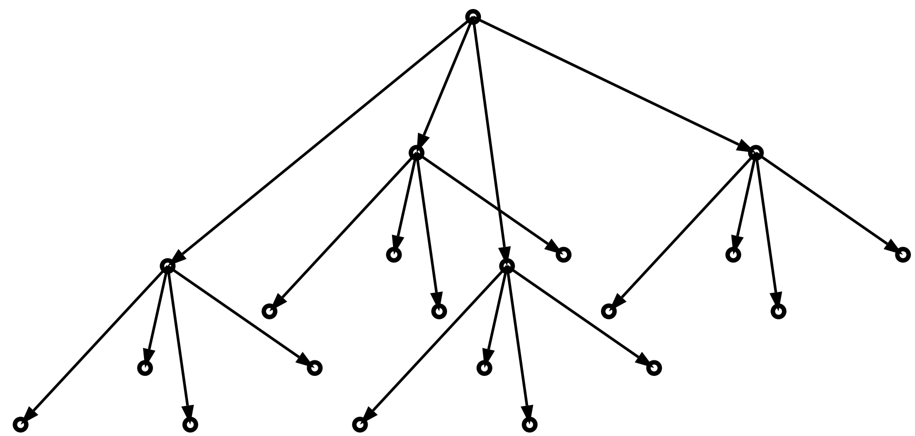
$$l_s^{(k)}(m) = \sum_{r \in d^{(k)}(s)} l(y_r|m)$$

  - Coarse scale statistics

$$t_1^{(k)} \triangleq t_1\left(x^{(k)}\right) \qquad t_2^{(k)} \triangleq t_2\left(x^{(k)}\right)$$

Recursions for Likelihood Function

- Organize blocks of image in quadtree structure



- Let $d(s)$ denote the four children of s, then

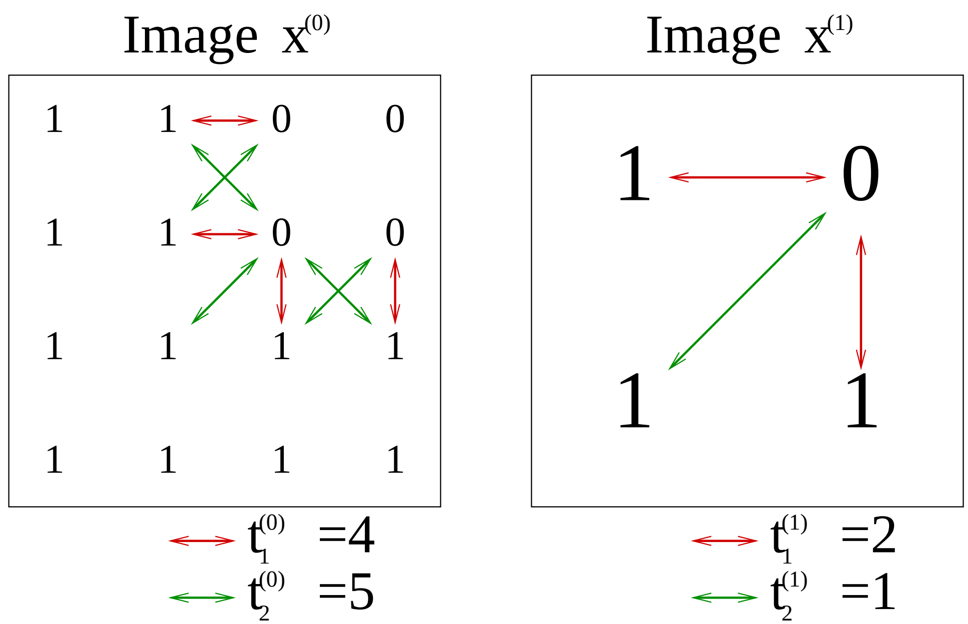
$$l_s^{(k)}(m) = \sum_{r \in d(s)} l_r^{(k-1)}(m)$$

where $l_s^{(0)}(m) = l(y_s|m)$.

- Complexity of recursion is order $\mathcal{O}(N)$ for $N = \#$ of pixels

Recursions for MRF Statistics

- Count statistics at each scale



- If x is 2^k-block-constant, then

$$
\begin{aligned}
t_1^{(k-1)} &= 2t_1^{(k)} \\
t_2^{(k-1)} &= 2t_1^{(k)} + t_2^{(k)}
\end{aligned}
$$

Parameter Scale Recursion [3]

- Assume x is 2^k-block-constant. Then we would like to select parameters $\beta_1^{(k)}$ and $\beta_2^{(k)}$ so that the energy functions match at each scale.

  This means that

$$\beta_1^{(k)} t_1^{(k)} + \beta_2^{(k)} t_2^{(k)} = \beta_1^{(k-1)} t_1^{(k-1)} + \beta_2^{(k-1)} t_2^{(k-1)}$$

- Substituting the recursions for $t_1^{(k)}$ and $t_2^{(k)}$ yeilds recursions for the parameters $\beta_1^{(k)}$ and $\beta_2^{(k)}$.

$$\begin{aligned} \beta_1^{(k)} &= 2\left(\beta_1^{(k-1)} + \beta_2^{(k-1)}\right) \\ \beta_2^{(k)} &= \beta_2^{(k-1)} \end{aligned}$$

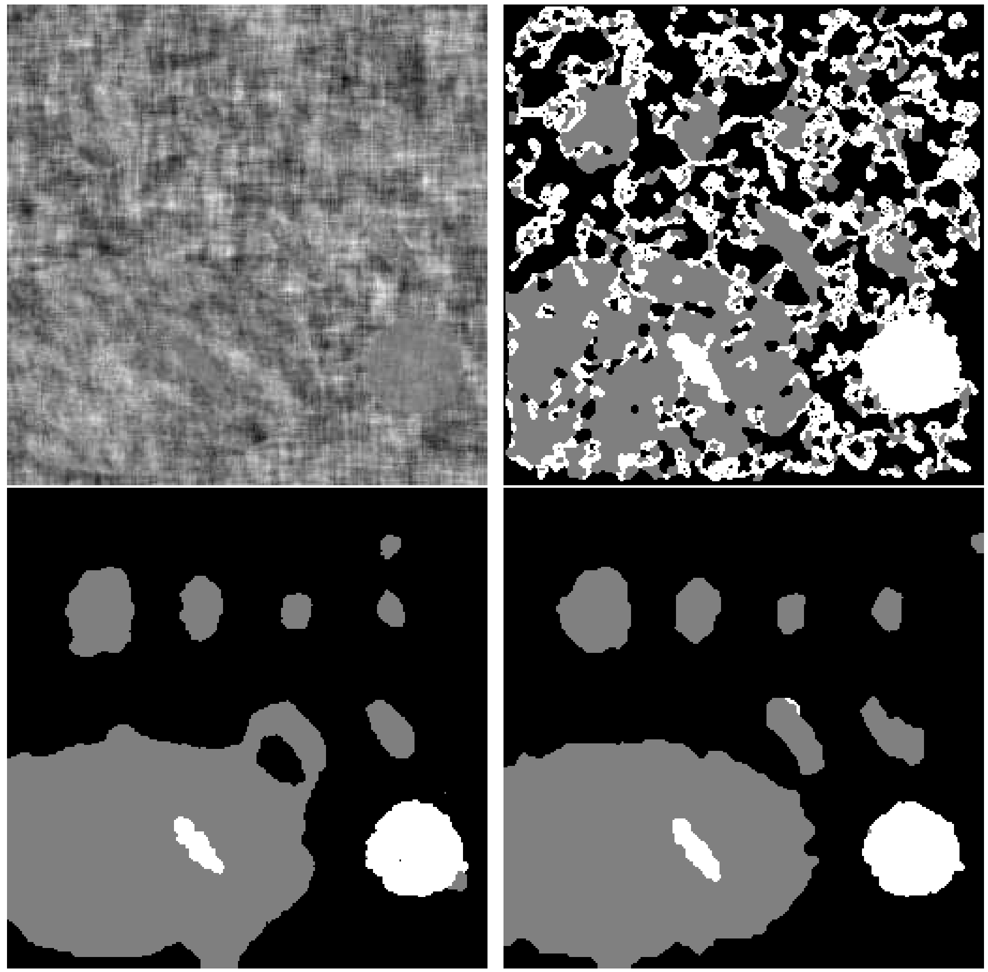
- Courser scale $\Rightarrow$ large β $\Rightarrow$ more smoothing
- Alternative approach: Leave β's constant

Multiple Resolution Segmentation (MRS) [3]

MRS Algorithm:

1. Select coarsest scale L and parameters $\beta_1^{(k)}$ and $\beta_2^{(k)}$
2. Set $l_s^{(0)}(m) \leftarrow l(y_s|m)$.
3. For $k = 1$ to L, compute: $l_s^{(k)}(m) = \sum_{r \in d(s)} l_r^{(k-1)}(m)$
4. Compute ML estimate at scale L: $\hat{x}_s^{(L)} \leftarrow \arg\min_{0 \le m < M} l_s^{(L)}(m)$
5. For $k = L$ to 0
   (a) Perform ICM optimization using inital condition $\hat{x}_s^{(L)}$ until converged
   $$\hat{x}^{(k)} \leftarrow ICM\left(\hat{x}^{(k)}, u^{(k)}(\cdot)\right)$$
   where
   $$u^{(k)}\left(\hat{x}^{(k)}\right) = \sum_s l_s^{(k)}(\hat{x}_s^{(k)}) + \beta_2^{(k)} t_1^{(k)} + \beta_2^{(k)} t_2^{(k)}$$
   (b) if $k > 0$ compute initial condition using block replication
   $$\hat{x}^{(k-1)} \leftarrow Block\_Replication(\hat{x}^{(k)})$$
6. Output $\hat{x}^{(0)}$

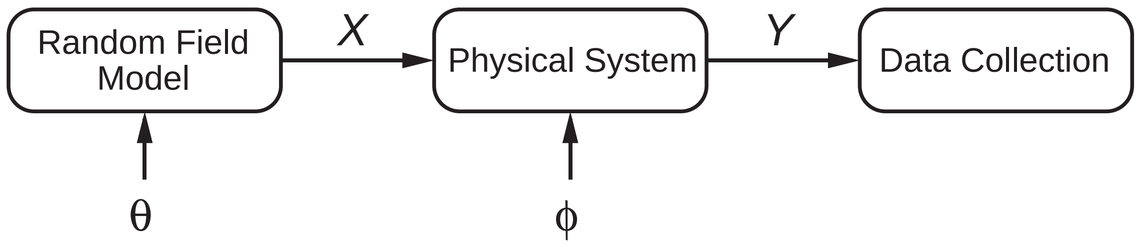
Texture Segmentation Example



a	b
c	d

a) Synthetic image with 3 textures b) ICM - 29 iterations c) Simulated Annealing - 100 iterations d) Multiresolution - 7.8 iterations

Parameter Estimation



- Question: How do we estimate θ from Y?
- Problem: We don't know X!
- Solution 1: Joint MAP estimation [11]

$$(\hat{\theta}, \hat{x}) = \arg\max_{\theta, x} p(y, x|\theta)$$

  – Problem: The solution is biased.

- Solution 2: Expectation maximization algorithm [1, 7]

$$\hat{\theta}^{k+1} = \arg\max_{\theta} E[\log p(Y, X|\theta)|Y = y, \theta^k]$$

  – Expectation may be computed using simulation techniques or mean field theory.

References

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