

# Lecture 5: Discrete-Time Random Processes

## Chapter 2.4

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## ① Random processes and their specification

Finite-dimensional distributions, the Kolmogorov extension theorem, and Gaussian random processes.

## ② Stationarity

Strict-sense and wide-sense stationarity. Mean, autocovariance, and autocorrelation.

## ③ The Gaussian simplification

Mean and autocovariance specify everything. The power spectral density and the Wiener–Khinchin theorem. Reversibility.

## ④ Two dimensions

Random fields, 2-D stationarity, and the 2-D power spectral density.

Reading: Chapter 2.4, *Foundations of Computational Imaging*.

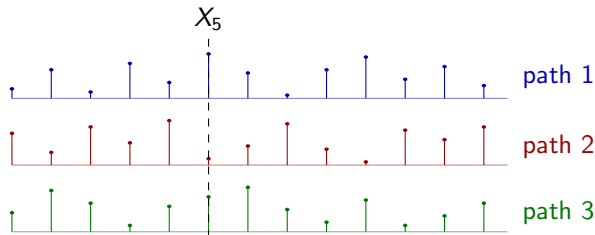
# Today's Big Ideas

- A random process is an infinite collection of random variables, so a whole signal or image becomes a single random object.
- The distribution of the process is specified by the distributions of its finite windows.
- Stationarity means the distribution of the process does not change when the process is shifted in time.
- A Gaussian process is completely specified by its mean and autocovariance. If it is also zero mean and stationary, the power spectral density alone specifies it.

In 2-D these same models describe images. We use them for the rest of the course.

# Discrete-time random processes

A **discrete-time random process**  $X_n$  is a random variable defined for each integer  $n \in \mathbb{Z}$ . Each run of the experiment produces an entire sample path:



- At each fixed time  $n$ , the value  $X_n$  is one random variable. It varies from path to path.

## The specification problem

A random process is an infinite collection of random variables. What does it take to specify its distribution?

# Finite-dimensional distributions

The **finite-dimensional distributions** of a random process are the joint distributions of its finite windows,

$$F_{m,k}(t_m, \dots, t_k) = P\{X_m \leq t_m, X_{m+1} \leq t_{m+1}, \dots, X_k \leq t_k\},$$

where  $m$  and  $k$  are any integers with  $m \leq k$ .

- These distributions are clearly necessary to specify the process.
- The next theorem says they are also sufficient.

# The Kolmogorov extension theorem

## Kolmogorov extension theorem

If the set of finite-dimensional distributions is consistent for all  $m, k \in \mathbb{Z}$ , then the distribution of the infinite-dimensional random process is well defined.

- In other words, a consistent set of finite windows determines the distribution of the whole infinite process.
- See, for example, Wong and Hajek, *Stochastic Processes in Engineering Systems*, page 3.

# Gaussian random processes

## Definition

A discrete-time random process  $X_n$  is a **Gaussian random process** if for all  $m \leq k$  the vector

$$[X_m, X_{m+1}, \dots, X_k]^T$$

is jointly Gaussian.

- This is a statement about every finite subset of the process, not about each  $X_n$  by itself. Recall from Lecture 3 that jointly Gaussian is stronger than Gaussian marginals.

# Strict-sense stationary random process

It is often reasonable to assume that shifting a random process in time does not change its distribution. This form of time invariance is called **stationarity**.

## Definition: Strict-Sense Stationary Random Process

A 1-D discrete-time random process  $X_n$  is said to be *strict-sense stationary* if  $X_n$  and its shifted version  $Y_n = X_{n-k}$  have the same finite-dimensional distributions for all integers  $k$ . This means that

$$F_{m,m+k}(t_0, t_1, \dots, t_k) = F_{0,k}(t_0, t_1, \dots, t_k)$$

for all integers  $m$  and  $k \geq 1$  and for all values  $t_0, t_1, \dots, t_k \in \mathbb{R}$ .

# Second-order random processes

Strict-sense stationarity is a strong condition. In practice it is much easier to check the second-order properties of a random process.

## Definition: Second-Order Random Process

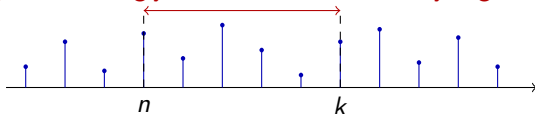
A 1-D discrete-time random process  $X_n$  is said to be a *second-order random process* if for all  $n, k \in \mathbb{Z}$ ,  $\mathbb{E}[|X_n X_k|] < \infty$ , that is all second-order moments are finite.

# Mean, autocovariance, and autocorrelation

If a random process is second order, then it has a well-defined mean  $\mu_n$  and **time autocovariance**  $R(n, k)$ :

$$\begin{aligned}\mu_n &= \mathbb{E}[X_n] , \\ R(n, k) &= \mathbb{E}[(X_n - \mu_n)(X_k - \mu_k)] .\end{aligned}$$

$R(n, k)$ : how strongly do these two values vary together?



We will also often talk about the **time autocorrelation**

$$C(n, k) = \mathbb{E}[X_n X_k].$$

- The autocorrelation and autocovariance are the same for a zero-mean random process.

# Wide-sense stationary random process

## Definition: Wide-Sense Stationary Random Process

A 1-D second-order discrete-time random process  $X_n$  is said to be *wide-sense stationary* if for all  $n, k \in \mathbb{Z}$ ,

$$\mathbb{E}[X_n] = \mu \quad \text{and} \quad \mathbb{E}[(X_n - \mu)(X_k - \mu)] = R(n - k) ,$$

where  $\mu$  is a scalar constant and  $R(n)$  is a scalar function of  $n$ .

- So a discrete-time random process is wide-sense stationary if its mean is constant and its time autocovariance is a function of only  $n - k$ , the difference between the two times.
- Strict-sense stationarity constrains all the statistics. Wide-sense stationarity constrains only two.

# Gaussian processes are specified by mean and autocovariance

In general, specifying a random process requires every finite-dimensional distribution. For a Gaussian process, the mean and the autocovariance are enough. This is the main reason Gaussian models are used so widely.

## Property: Autocovariance specification of Gaussian random processes

The distribution of a discrete-time Gaussian random process  $X_n$  is uniquely specified by its mean  $\mu_n$  and time autocovariance  $R(n, k)$ .

- Every finite window of the process is jointly Gaussian, and a Gaussian vector is determined by its mean and covariance. Both are read from  $\mu_n$  and  $R(n, k)$ .

# Power spectral density: the Wiener–Khinchin theorem

- If a Gaussian random process is also stationary, then we can equivalently specify its distribution from its **power spectral density**  $S_X(e^{j\omega})$ .

## Wiener–Khinchin theorem

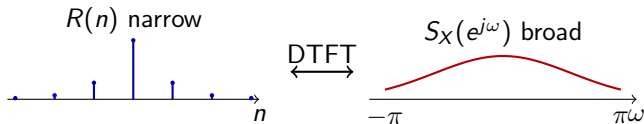
For a zero-mean wide-sense stationary random process, the power spectral density is given by

$$S_X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R(n) e^{-j\omega n} ,$$

the **discrete-time Fourier transform (DTFT)** of the time autocovariance  $R(n)$ .

# The power spectral density specifies the distribution

- Since the DTFT is an invertible transform, knowing  $S_X(e^{j\omega})$  is equivalent to knowing the time autocovariance  $R(n)$ .



- A narrow autocovariance describes a fast-changing process, and it spreads power across all frequencies. A broad autocovariance describes a smooth process, and it concentrates power at low frequencies.

## Property: Power spectral density specification

The distribution of a zero-mean wide-sense stationary discrete-time Gaussian random process  $X_n$  is uniquely specified by its power spectral density  $S_X(e^{j\omega})$ .

# Reversible random process

A random process is **reversible** if it has the same distribution as its time reverse  $X_{-n}$ . Reversibility is a second invariance property, alongside stationarity.

## Definition: Reversible Random Process

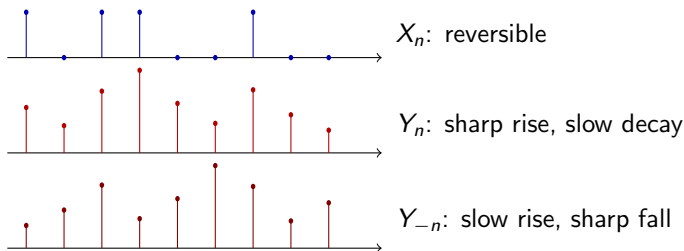
A 1-D discrete-time random process  $X_n$  is said to be *reversible* if its finite-dimensional distributions have the property that

$$F_{0,k}(t_0, t_1, \dots, t_k) = F_{m,m+k}(t_k, t_{k-1}, \dots, t_0)$$

for all integers  $m$  and  $k \geq 1$  and for all values  $t_0, t_1, \dots, t_k \in \mathbb{R}$ .

# Not all random processes are reversible

Let  $X_n$  be an i.i.d. Bernoulli process, which is a random binary sequence. It is stationary and reversible, because it has the same distribution run forward and run backward. Now filter it with a causal IIR filter.



- The filtered process  $Y_n$  is still stationary, but its time reverse has a different distribution. So  $Y_n$  is not reversible.

# Reversibility of Gaussian random processes

Let  $X_n$  be a wide-sense stationary Gaussian random process, and let  $Y_n = X_{-n}$  be its time reverse. By the definition of time autocovariance,

$$\begin{aligned} R_Y(n) &= \mathbb{E}[(Y_k - \mu)(Y_{k+n} - \mu)] \\ &= \mathbb{E}[(X_{-k} - \mu)(X_{-k-n} - \mu)] \\ &= \mathbb{E}[(X_{l+n} - \mu)(X_l - \mu)] \\ &= R_X(n), \end{aligned}$$

where  $l = -k - n$ , and  $\mu$  is the mean of both  $X_n$  and  $Y_n$ . So  $X_n$  and  $Y_n$  have the same autocovariance, and by the autocovariance specification property they have the same distribution.

## Property: Reversibility of stationary Gaussian random processes

Any wide-sense stationary Gaussian random process is also strict-sense stationary and reversible.

# Generalization to 2-D

Strict-sense stationarity, wide-sense stationarity, autocovariance, power spectral density, and reversibility all generalize naturally to 2-D and higher dimensions.

Let  $X_{s_1, s_2}$  be a 2-D discrete-space random process that is wide-sense stationary. Its 2-D autocovariance function is

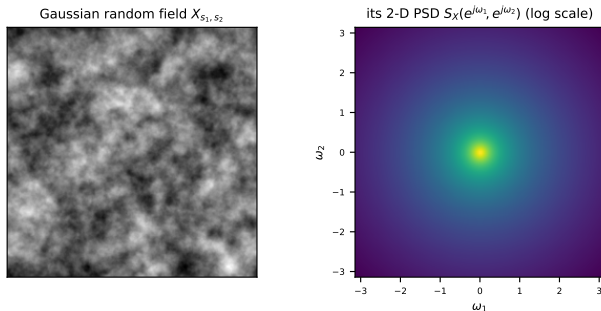
$$R(s_1 - r_1, s_2 - r_2) = \mathbb{E}[(X_{s_1, s_2} - \mu)(X_{r_1, r_2} - \mu)] ,$$

where  $\mu = \mathbb{E}[X_{s_1, s_2}]$ . Its 2-D power spectral density is

$$S_X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} R(s_1, s_2) e^{-j\omega_1 s_1 - j\omega_2 s_2} ,$$

the **discrete-space Fourier transform (DSFT)** of the 2-D autocovariance.

# A stationary Gaussian random field



- Left: one realization of a 2-D stationary Gaussian random process, called a **random field**. Any shifted window of the image has the same distribution, which is what 2-D stationarity means.
- Right: its power spectral density. Power concentrated at low spatial frequencies produces the smooth texture.
- This is our first image-like random object. The image models of the coming chapters build directly on it.

# Stationarity and reversibility in 2-D

- The 2-D random process is said to be **stationary** if  $X_{s_1, s_2}$  and  $Y_{s_1, s_2} = X_{s_1 - k_1, s_2 - k_2}$  have the same distribution for all integers  $k_1$  and  $k_2$ .
- The 2-D random process is said to be **reversible** if  $X_{s_1, s_2}$  and  $Y_{s_1, s_2} = X_{-s_1, -s_2}$  have the same distribution.

Again, it can be shown that 2-D zero-mean wide-sense stationary Gaussian random processes are always reversible and strict-sense stationary.

# Summary

- A random process is an infinite collection of random variables  $X_n$ . Its distribution is specified by its finite-dimensional distributions  $F_{m,k}(t_m, \dots, t_k) = P\{X_m \leq t_m, \dots, X_k \leq t_k\}$ , and by the Kolmogorov extension theorem those are enough.
- Strict-sense stationarity means that no finite-dimensional distribution changes under a shift,  $F_{m,m+k} = F_{0,k}$ .
- Wide-sense stationarity constrains only the first two moments,  $\mathbb{E}[X_n] = \mu$  and  $\mathbb{E}[(X_n - \mu)(X_k - \mu)] = R(n - k)$ .
- A Gaussian random process is completely specified by its mean  $\mu_n$  and its time autocovariance  $R(n, k)$ .
- If it is also zero mean and wide-sense stationary, the power spectral density  $S_X(e^{j\omega}) = \sum_n R(n)e^{-j\omega n}$  specifies it. This is the Wiener–Khinchin theorem.
- Such a process is also strict-sense stationary and reversible. In 2-D the same results hold, with the DSFT of the 2-D autocovariance.

**Reading:** Chapter 2.4.    **Next lecture:** Chapter 3.1–3.2.