EE 438 Essential Definitions and Relations

CT Signals and Operators

1. $\text{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$

2. $\text{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$

3. $\delta(t) = \lim\limits_{\Delta \to 0} \dfrac{1}{\Delta} \text{rect}\left(\dfrac{t}{\Delta}\right)$

4. $x(t) \;\; y(t) = \int_{-\infty}^{\infty} x(t-\tau) y(\tau) d\tau$

5. $\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$

6. $\text{comb}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$

Properties of the CT Impulse

1. $\int_{-\infty}^{\infty} \delta(t) dt = 1$ and $\delta(t) = 0, \; t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t - t_0)\, x(t) dt = x(t_0)$,

   provided $x(t)$ is continuous at $t = t_0$

3. $\delta(t - t_0) \;\; x(t) = x(t - t_0)$

CTFT

Sufficient conditions for existence

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \text{ or } \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Forward transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} df$$

Inverse transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} dt$$

Transform Relations

1. Linearity

$$a_1 x_1(t) + a_2 x_2(t) \overset{\text{CTFT}}{\leftrightarrow} a_1 X_1(f) + a_2 X_2(f)$$

2. Scaling and shifting

$$x\left(\frac{t - t_0}{a}\right) \overset{\text{CTFT}}{\leftrightarrow} |a| X(af) e^{-j2\pi f t_0}$$

3. Modulation

$$x(t) e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} X(f - f_0)$$

4. Reciprocity

$$X(t) \overset{\text{CTFT}}{\leftrightarrow} x(-f)$$

5. Parseval

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

6. Initial Value

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

$$x(0) = \int_{-\infty}^{\infty} X(f) df$$

7. Convolution

$$x(t) \;\; y(t) \overset{\text{CTFT}}{\leftrightarrow} X(f) Y(f)$$

8. Product

$$x(t) y(t) \overset{\text{CTFT}}{\leftrightarrow} X(f) \;\; Y(f)$$

9. Transform of periodic signal

$$\text{rep}_T[x(t)] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}[X(f)]$$

10. Transform of sampled signal

$$\text{comb}_{\text{T}}[x(t)] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{\text{T}} \text{rep}_{\frac{1}{\text{T}}}[X(f)]$$

Transform Pairs

1. $\text{rect(t)} \overset{\text{CTFT}}{\leftrightarrow} \text{sinc(f)}$

2. $\delta(t) \overset{\text{CTFT}}{\leftrightarrow} 1$ and $1 \overset{\text{CTFT}}{\leftrightarrow} \delta(f)$

3. $e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} \delta(f - f_0)$

4. $\cos(2\pi f_0 t) \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$

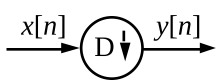
DT Signals and Operators

1. $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

2. $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

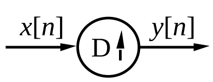
3. $x[n] \;\; y[n] = \sum_{k=-\infty}^{\infty} x[n-k]y[k]$

4. Downsampling



$$y[n] = x[Dn]$$

5. Upsampling



$$y[n] = \begin{cases} x[n/D], & n/D \text{ integer - valued} \\ 0, & \text{else} \end{cases}$$

Properties of the DT Impulse

1. $\sum_{n=-\infty}^{\infty} \delta[n] = 1$ and $\delta[n] = 0, n \neq 0$

2. $\sum_{n=-\infty}^{\infty} \delta[n - n_0] x[n] = x[n_0]$

3. $\delta[n - n_0] \;\; x[n] = x[n - n_0]$

DTFT

Sufficient conditions for existence

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{or} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Forward transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Transform Relations

1. Linearity

$$a_1 x_1[n] + a_2 x_2[n] \overset{\text{DTFT}}{\leftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega)$$

2. Shifting

$$x[n - n_0] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) e^{-j\omega n_0}$$

3. Modulation

$$x[n] e^{j\omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} X(\omega - \omega_0)$$

4. Parseval

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

5. Initial Value

$$X(0) = \sum_{n=-\infty}^{\infty} x[n]$$

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) d\omega$$

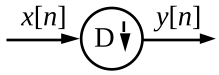
6. Convolution

$$x[n] \;\; y[n] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) Y(\omega)$$

7. Product

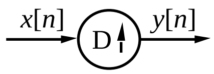
$$x[n]y[n] \overset{\text{DTFT}}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\mu)Y(\mu)\,d\mu$$

8. Downsampling



$$Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right)$$

10. Upsampling



$$Y(\omega) = X(D\omega)$$

Transform Pairs

1. $\delta[n] \overset{\text{DTFT}}{\leftrightarrow} 1$

2. $1 \overset{\text{DTFT}}{\leftrightarrow} 2\pi\,\mathrm{rep}_{2\pi}[\delta(\omega)]$

3. $e^{j\omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} 2\pi\,\mathrm{rep}_{2\pi}[\delta(\omega-\omega_0)]$

4. $\cos(\omega_0 n) \overset{\text{DTFT}}{\leftrightarrow}$
$\pi\,\mathrm{rep}_{2\pi}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

Relation between CT and DT

1. Time Domain

$$x_{\mathrm{d}}[n] = x_{\mathrm{a}}(nT)$$

2. Frequency Domain

$$X_{\mathrm{d}}(\omega_{\mathrm{d}}) = f_{\mathrm{s}}\mathrm{rep}_{f_{\mathrm{s}}}\left[X_{\mathrm{a}}(f_{\mathrm{a}})\right]\Big|_{f_{\mathrm{a}}=\frac{\omega_{\mathrm{d}} f_{\mathrm{s}}}{2\pi}},$$

where $f_{\mathrm{s}} = \dfrac{1}{T}$

ZT

Forward transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Transform Relations

1. Linearity

$$a_1x_1[n] + a_2x_2[n] \overset{\text{ZT}}{\leftrightarrow} a_1X_1(z) + a_2X_2(z)$$

2. Shifting

$$x[n-n_0] \overset{\text{ZT}}{\leftrightarrow} X(\omega)z^{-n_0}$$

3. Modulation

$$x[n]z_0^n \overset{\text{ZT}}{\leftrightarrow} X(z/z_0)$$

4. Multiplication by time index

$$nx[n] \overset{\text{ZT}}{\leftrightarrow} -z\frac{d}{dz}[X(z)]$$

6. Convolution

$$x[n]\ \ y[n] \overset{\text{ZT}}{\leftrightarrow} X(z)Y(z)$$

7. Relation to DTFT

If $X_{\mathrm{ZT}}(z)$ converges for $|z|=1$, then the DTFT of $x[n]$ exists and

$$X_{\mathrm{DTFT}}(\omega) = X_{\mathrm{ZT}}(z)\Big|_{z=e^{j\omega}}$$

Transform Pairs

1. $\delta[n] \overset{\text{ZT}}{\leftrightarrow} 1, \quad \text{all } z$

2. $a^n u[n] \overset{\text{ZT}}{\leftrightarrow} \dfrac{1}{1-az^{-1}}, \quad |z|>|a|$

3. $-a^n u[-n-1] \overset{\text{ZT}}{\leftrightarrow} \dfrac{1}{1-az^{-1}}, \quad |z|<|a|$

4. $na^n u[n] \overset{\text{ZT}}{\leftrightarrow} \dfrac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$

5. $-na^n u[-n-1] \overset{\text{ZT}}{\leftrightarrow} \dfrac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z|<|a|$

DFT

Forward transform

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

Inverse transform

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

Transform Relations

1. Linearity

$$a_1x_1[n] + a_2x_2[n] \overset{\text{DFT}}{\leftrightarrow} a_1X_1[k] + a_2X_2[k]$$

2. Shifting

$$x[n-n_0] \overset{\text{DFT}}{\leftrightarrow} X[k]e^{-j2\pi n_0 k/N}$$

3. Modulation

$$x[n]e^{j2\pi k_0/N} \overset{\text{DFT}}{\leftrightarrow} X[k-k_0]$$

4. Reciprocity

$$X[n] \overset{\text{DFT}}{\leftrightarrow} N\,x[-k]$$

5. Parseval

$$\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

6. Initial Value

$$X[0] = \sum_{n=0}^{N-1} x[n]$$

$$x[0] = \sum_{k=0}^{N-1} X[k]$$

7. Periodicity

$x[n+mN] = x[n]$ for all integers m

$X[k+lN] = X[k]$ for all integers l

8. Relation to DTFT of a finite length sequence

Let $x_0[n] \neq 0$ only for $0 \le n \le N-1$

$$x_0[n] \overset{\text{DTFT}}{\leftrightarrow} X_0(\omega).$$

Define $x[n] = \sum_{m=-\infty}^{\infty} x_0[n+mN]$,

$$x[n] \overset{\text{DFT}}{\leftrightarrow} X[k],$$

then $X[k] = X_0\left(e^{j2\pi k/N}\right)$.

6. Periodic convolution

$$x[n] \bigcirc y[n] \overset{\text{DFT}}{\leftrightarrow} X[k]Y[k]$$

7. Product

$$x[n]y[n] \overset{\text{DFT}}{\leftrightarrow} X[k] \bigcirc Y[k]$$

Transform Pairs

1. $\delta[n], 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$ $1, 0 \le k \le N-1$

2. $1, 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$ $N\delta[k], 0 \le k \le N-1$

3. $e^{j2\pi k_0 n/N}, 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$ $N\delta[k-k_0], 0 \le k \le N-1$

4. $\cos(2\pi k_0 n/N), 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$ $\frac{N}{2}\{\delta[k-k_0] + \delta[k-(N-k_0)]\},$ $0 \le k \le N-1$

Random Signals

1. Probability density function $f_X(x)$

$$P\{a < X \le b\} = \int_a^b f_X(x)dx$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad f_X(x) \ge 0$$

2. Probability distribution function $F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$F_X(-\infty) = 0 \quad F_X(\infty) = 1 \quad F_X(x)$ is a nondecreasing function of x

Expectation Operator

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

1. Mean $g(x) = x$

2. 2nd moment $g(x) = x^2$

3. Variance $g(x) = (x - E\{X\})^2$

4. Relation between mean, 2nd moment, and variance

$$\sigma_X^2 = E\{X^2\} - (E\{X\})^2$$

Two Random Variables

1. Bivariate probability density function $f_{XY}(x,y)$

$$P\{a < X \leq b, c < Y \leq d\} = \int_a^b \int_c^d f_{XY}(x,y)dxdy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1 \quad f_{XY}(x,y) \geq 0$$

2. Marginal probability density functions $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

3. Linearity of expectation

$$E\{ag(X,Y) + bh(X,Y)\} = aE\{g(X,Y)\} + bE\{h(X,Y)\}$$

4. Correlation of X and Y

$$E\{XY\} = \int_a^b \int_c^d xy f_{XY}(x,y)dxdy$$

5. Covariance of X and Y

$$\sigma_{XY}^2 = E\{(X - E\{X\})(Y - E\{Y\})\}$$
$$= E\{XY\} - E\{X\}E\{Y\}$$

6. Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$

$$0 \leq |\rho_{XY}| \leq 1$$

7. X and Y are independent if and only if

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

8. X and Y are uncorrelated if and only if

$$E\{XY\} = E\{X\}E\{Y\}$$

Independence implies uncorrelatedness. The converse is not true.

Discrete-time random signals

1. Autocorrelation function

$$r_{XX}[m, m+n] = E\{X[m]X[m+n]\}$$

$X[m]$ is wide-sense stationary if and only if mean is constant, and autocorrelation depends only on difference between arguments, i.e.

$$E\{X[n]\} = \mu_X$$

$$r_{XX}[n] = E\{X[m]X[m+n]\}$$

2. Cross-correlation function

$$r_{XY}[m, m+n] = E\{X[m]Y[m+n]\}$$

$X[m]$ and $Y[m]$ are jointly wide-sense stationary if and only if they are individually wide-sense stationary and their cross-correlation depends only on difference between arguments, i.e.

$$r_{XY}[n] = E\{X[m]Y[m+n]\}$$

3. Wide-sense stationary processes and linear systems.

Let $X[n]$ be wide-sense stationary, and suppose that

$$Y[n] = h[n] \;\; X[n] = \sum_m h[n-m]X[m],$$

then $X[n]$ and $Y[n]$ are jointly wide-sense stationary, and

$$\mu_X = \mu_Y \sum_m h[m]$$

$$r_{XY}[n] = h[n] \;\; r_{XX}[n]$$

$$r_{YY}[n] = h[n] \;\; r_{XY}[-n]$$

2D CS Signals and Operators

1. $\mathrm{rect}(x,y) = \begin{cases} 1, & |x|,|y| < 1/2 \\ 0, & |x|,|y| > 1/2 \end{cases}$

2. $\mathrm{sinc}(x,y) = \dfrac{\sin(\pi x)}{\pi x}\dfrac{\sin(\pi y)}{\pi y}$

3. $\mathrm{circ}(x,y) = \begin{cases} 1, & \sqrt{x^2+y^2} < 1/2 \\ 0 & \sqrt{x^2+y^2} > 1/2 \end{cases}$

4. $\mathrm{jinc}(x,y) = \dfrac{J_1\left(\pi\sqrt{x^2+y^2}\right)}{2\sqrt{x^2+y^2}}$

5. $\delta(x,y) = \lim\limits_{\Delta\to 0}\dfrac{1}{\Delta^2}\,\mathrm{rect}\left(\dfrac{x}{\Delta},\dfrac{y}{\Delta}\right)$

6. $f(x,y) \;\; g(x,y) =$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)g(\xi,\eta)\,d\xi\, d\eta$$

7. $\mathrm{rep}_{XY}[f(x,y)] =$
$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f(x-kX, y-lY)$$

8. $\mathrm{comb}_{XY}[f(x,y)] =$
$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f(kX,lY)\delta(x-kX, y-lY)$$

9. Separability

$$f(x,y) = g(x)\,h(y)$$

Properties of the CS Impulse

1. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)\,dx\,dy = 1$$
and $\delta(x,y) = 0,\ (x,y) \neq (0,0)$

2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-x_0, y-y_0)f(x,y)\,dx\,dy = f(x_0,y_0),$$
provided $f(x,y)$ is continuous at $(x,y) = (x_0,y_0)$

3. $\delta(x-x_0, y-y_0) \;\; f(x,y) = f(x-x_0, y-y_0)$

CSFT

Sufficient conditions for existence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x,y)|\,dx\,dy < \infty$$

or $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x,y)|^2\,dx\,dy < \infty$$

Forward transform

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

Inverse transform

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$

Transform Relations

1. Linearity

$$a_1 f_1(x,y) + a_2 f_2(x,y) \overset{\text{CSFT}}{\leftrightarrow} a_1F_1(u,v) + a_2F_2(u,v)$$

2. Scaling and shifting

$$f\left(\frac{x-x_0}{a},\frac{y-y_0}{b}\right) \overset{\text{CSFT}}{\leftrightarrow}$$

$$|ab|\,F(au,bv)e^{-j2\pi(ux_0+vy_0)}$$

3. Modulation

$$f(x,y)e^{j2\pi(u_0x+v_0y)} \overset{\text{CSFT}}{\leftrightarrow} F(u-u_0,v-v_0)$$

4. Reciprocity

$$F(x,y) \overset{\text{CSFT}}{\leftrightarrow} f(-u,-v)$$

5. Parseval

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x,y)|^2\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(u,v)|^2\,\mathrm{d}u\mathrm{d}v$$

6. Initial Value

$$F(0,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$\mathrm{x}(0,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\,\mathrm{d}u\mathrm{d}v$$

7. Convolution

$$f(x,y) \quad g(x,y) \overset{\text{CSFT}}{\leftrightarrow} F(u,v)G(u,v)$$

8. Product

$$f(x,y)g(x,y) \overset{\text{CSFT}}{\leftrightarrow} F(u,v) \quad G(u,v)$$

9. Transform of doubly periodic signal

$$\mathrm{rep}_{XY}[f(x,y)] \overset{\text{CSFT}}{\leftrightarrow} \frac{1}{XY}\mathrm{comb}_{\frac{1}{XY}}[F(u,v)]$$

10. Transform of sampled signal

$$\mathrm{comb}_{XY}[f(x,y)] \overset{\text{CSFT}}{\leftrightarrow} \frac{1}{XY}\mathrm{rep}_{\frac{1}{XY}}[F(u,v)]$$

11. Transform of separable signal

$$g(x)\,h(y) \overset{\text{CSFT}}{\leftrightarrow} G(u)\,H(v)$$

Transform Pairs

1. $\mathrm{rect}(x,y) \overset{\text{CSFT}}{\leftrightarrow} \mathrm{sinc}(u,v)$

2. $\mathrm{circ}(x,y) \overset{\text{CSFT}}{\leftrightarrow} \mathrm{jinc}(u,v)$

3. $\delta(x,y) \overset{\text{CSFT}}{\leftrightarrow} 1$ and $1 \overset{\text{CSFT}}{\leftrightarrow} \delta(u,v)$

4. $e^{j2\pi(u_0x+v_0y)} \overset{\text{CSFT}}{\leftrightarrow} \delta(u-u_0,v-v_0)$

5.
$$\cos\left[2\pi(u_0x+v_0y)\right] \overset{\text{CTFT}}{\leftrightarrow}$$

$$\frac{1}{2}\{\delta(u-u_0,v-v_0)+\delta(\delta(u+u_0,v+v_0))\}$$

Computed Tomography

1. Radon transform

$$p_\theta(t) = \iint g(x,y)\delta(x\cos\theta + y\sin\theta - t)\mathrm{d}x\mathrm{d}y$$

2. Fourier Slice Theorem

$$P_\theta(\rho) = G(\rho\cos\theta,\ \rho\sin\theta)$$

Filtered/Convolution Backprojection

1. Filtered projection

$$Q_\theta(\rho) = |\rho| P_\theta(\rho)$$

1. Filtering implemented by convolution

$$q_\theta(t) = h_B(t) \quad p_\theta(t)$$

where $h_B(t) \overset{\text{CTFT}}{\leftrightarrow} |\rho|\,\mathrm{rect}(\rho/\mathrm{B})$

2. Backprojection

$$g(x,y) = \int_0^\pi q_\theta(x\cos\theta + y\sin\theta)\mathrm{d}\theta$$

Miscellaneous

1. Geometric Series

$$\sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z}$$