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EE 438 DIGITAL SIGNAL PROCESSING WITH APPLICATIONS

Final Exam – Tuesday, May 4, 1999

- You have 110 minutes to complete the following SIX problems.
- It is to your advantage to budget your time so that you can try every problem.
- The examination is closed-book, closed-notes and open mind.
- You must show all work to obtain full credit.
- No calculators are allowed.

Good Luck!

Some useful formulas:

1-D Transforms

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$
$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$
$$e^{-\pi t^2} \overset{CTFT}{\Leftrightarrow} e^{-\pi f^2}$$
$$x(t/T) \overset{CTFT}{\Leftrightarrow} |T|X(fT)$$
$$x(t-d) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi fd}$$
$$x(t)e^{j2\pi f_o} \overset{CTFT}{\Leftrightarrow} X(f-f_o)$$

Sampling

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{2\pi T}\right)$$
$$S(f) = Y(e^{j2\pi fT})$$

Interpolation and Decimation

$$Z(e^{j\omega}) = Y(e^{jL\omega})$$
$$Z(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} Y\left(e^{j(\omega-2\pi k)/L}\right)$$

2-D Transforms

$$\mathrm{rect}(x,y) \overset{CSFT}{\Leftrightarrow} \mathrm{sinc}(u,v)$$
$$\mathrm{circ}(x,y) \overset{CSFT}{\Leftrightarrow} \mathrm{jinc}(u,v)$$
$$\mathrm{circ}(x,y) = \begin{cases} 1 & if\ \sqrt{x^2+y^2} < 1/2 \\ 0 & otherwise \end{cases}$$

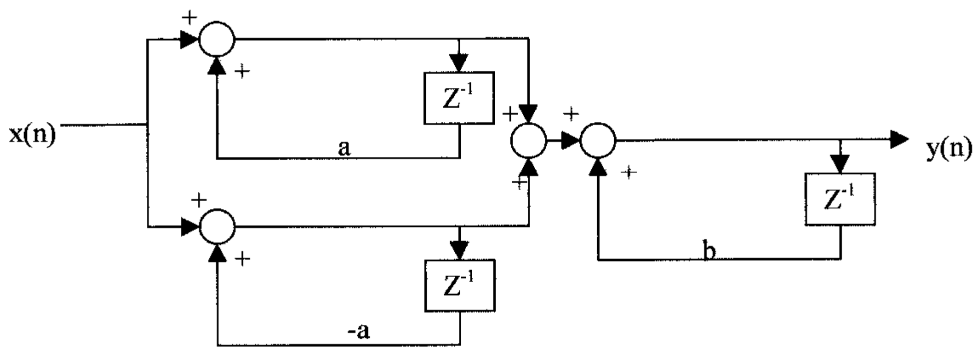
Z-Transforms

$$a^n u(n) \Leftrightarrow \frac{1}{1-az^{-1}} \quad ROC = |z| > a$$
$$-a^n u(-1-n) \Leftrightarrow \frac{1}{1-az^{-1}} \quad ROC = |z| < a$$

Name: ____________________

Problem 1. (33 points)

Consider the following discrete time system.



a) Calculate the transfer function $H(z)$ for the following system.

b) Calculate the difference equation for the system. c) Calculate the pole of the system.

d) Assuming that the system is causal, when is the difference equation stable?

a) $$H(z) = \left(\frac{1}{1-az^{-1}} + \frac{1}{1+az^{-1}}\right)\left(\frac{1}{1-bz^{-1}}\right)$$

$$= \frac{1+az^{-1}+1-az^{-1}}{(1-az^{-1})(1+az^{-1})(1-bz^{-1})}$$

$$= \frac{1}{(1-az^{-1})(1+az^{-1})(1-bz^{-1})}$$

$$= \frac{1}{(1-a^2z^{-2})(1-bz^{-1})}$$

$$= \frac{1}{1-bz^{-1}-a^2z^{-2}+a^2bz^{-3}}$$

b) $$(1-bz^{-1}-a^2z^{-2}+a^3bz^{-3})\,Y(z) = X(z)$$

$$y(n) = x(n) + b\,y(n-1) + a^2 y(n-2) + a^3 b\,y(n-3)$$

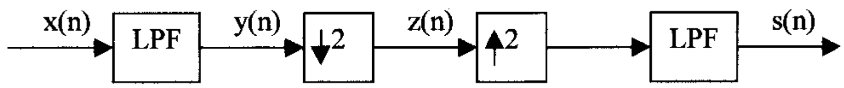
c) The poles of the system are $a, -a$ and b.

d) Causal system is stable $\Leftrightarrow$ poles are inside unit circle $\Leftrightarrow (|a|<1$ and $|b|<1)$

Name: ____________________

Problem 2. (33 points)

Consider the following discrete time system where $x(n) = \mathrm{sinc}(n/4)$ ↙ where the low pass filters have a cut off frequency of $\pi/2$ and a gain of 1.



a) Calculate $X(e^{j\omega})$.
b) Calculate $y(n)$ and $Y(e^{j\omega})$.
c) Calculate $z(n)$ and $Z(e^{j\omega})$.
d) Calculate $s(n)$ and $S(e^{j\omega})$.

a) $X(e^{j\omega}) = 4\,\mathrm{rect}(4\omega/(2\pi))$ for $|\omega| < \pi$

$= 4\,\mathrm{rect}(2\omega/\pi)$ for $|\omega| < \pi$

b) $H(e^{j\omega}) = \mathrm{rect}(\omega/\pi)$ for $|\omega| < \pi$

$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$

$= 4\,\mathrm{rect}(2\omega/\pi)\,\mathrm{rect}(\omega/\pi)$ for $|\omega| < \pi$

$= 4\,\mathrm{rect}(2\omega/\pi)$ for $|\omega| < \pi$

$\Rightarrow y(n) = \mathrm{sinc}(n/4)$

c) $z(n) = y(2n) = \mathrm{sinc}(2n/4)$

$= \mathrm{sinc}(n/2)$

$Z(e^{j\omega}) = 2\,\mathrm{rect}(\omega/\pi) \quad |\omega| < \pi$

d) $S(e^{j\omega}) = Z(e^{j2\omega})\,\mathrm{rect}(\omega/\pi) \quad |\omega| < \pi$

$$= \left[2\,\mathrm{rect}(2\omega/\pi) + 2\,\mathrm{rect}((2\omega - 2\pi)/\pi) - 2\,\mathrm{rect}((2\omega + 2\pi)/\pi) \right] \mathrm{rect}(\omega/\pi)$$

for $|\omega| < \pi$

$= 2\,\mathrm{rect}(2\omega/\pi)$

$s(n) = \frac{1}{2}\,\mathrm{sinc}(n/4)$

Name: ______________________

Problem 3. (33 points)

Compute the 8-point DFT of the following signals.

a) $x(n) = \begin{cases} 1 & \text{for } 0 \le n \le 2 \\ 0 & \text{for } 3 \le n \le 7 \end{cases}$

b) $y(n) = e^{j2\pi n/4}$ for $0 \le n \le 7$

c) $z(n) = \sum_{l=0}^{7} x(l)\, e^{2\pi(n-l)/4}$

a) $X(k) = \sum_{n=0}^{2} e^{-j2\pi k/8} = \dfrac{1 - e^{-j2\pi 3/8}}{1 - e^{-j2\pi/8}}$

$= \dfrac{e^{-j2\pi 3k/16}}{e^{-j2\pi k/16}} \dfrac{e^{j2\pi 3k/16} - e^{-j2\pi 3k/16}}{e^{+j2\pi k/16} - e^{-j2\pi k/16}}$

$= e^{-j2\pi k/8} \dfrac{\sin(2\pi 3k/16)}{\sin(2\pi k/16)}$

$= e^{-j2\pi k/8} \dfrac{\sin(3\pi k/8)}{\sin(\pi k/8)}$

b) $Y(k) = \sum_{n=0}^{7} e^{j2\pi n/4}\, e^{-j2\pi nk/8}$

$= \sum_{n=0}^{7} e^{j2\pi n(2-k)/8}$

for $k=2 \Rightarrow Y(k) = \sum_{n=0}^{7} 1 = 8$

for $k \neq 2 \Rightarrow Y(k) = \dfrac{1 - e^{-j2\pi 8(2-k)/8}}{1 - e^{-j2\pi(2-k)/8}}$

$= \dfrac{1-1}{1 - e^{-j2\pi(2-k)/8}} = 0$

$Y(k) = 8\delta(k-2)$ for $k = 0, \dots, 7$

c) $z(n) = x(n) \circledast_8 y(n)$

periodic convolution

$$Z(k) = X(k) \cdot Y(k)$$

$$= 8\delta(k-2)\, e^{-j2\pi k/8} \frac{\sin(3\pi k/8)}{\sin(\pi k/8)}$$

$$= 8\delta(k-2)\, e^{-j2\pi/4} \frac{\sin(3\pi/4)}{\sin(\pi/4)}$$

$$\sin(\pi/4) = 1/\sqrt{2}$$

$$\sin(3\pi/4) = 1/\sqrt{2}$$

$$Z(k) = 8\delta(k-2)\, e^{-j2\pi/4}$$

Name: ______________________

Problem 4. (33 points)

Let the 1-D discrete time function $h(n)$ have the DTFT of $H(e^{j\omega}) = rect(\omega/\pi)$ for $|\omega| < \pi$ and let the 2-D discrete time function $f(m,n)$ have a DSFT of
$F(e^{j\mu}, e^{j\nu}) = rect(\mu/\pi, \nu/\pi) - rect(2\mu/\pi, 2\nu/\pi)$ for $|\mu| < \pi$ and $|\nu| < \pi$ ~~and let~~

a) Calculate $h(n)$.

b) Is $f(m,n)$ a separable function? Justify your answer.

c) Calculate $f(m,n)$.

d) Calculate $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)$.

a)
$$h(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\frac{1}{jn}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, jn\, d\omega$$
$$= \frac{1}{2\pi jn} e^{j\omega n}\Big|_{-\pi/2}^{\pi/2} = \frac{1}{2\pi jn}\left[e^{jn\pi/2} - e^{-jn\pi/2}\right]$$
$$= \frac{1}{\pi n}\sin(\pi n/2)$$
$$= \frac{1}{2}\frac{\sin(\pi n/2)}{\pi n/2} = \frac{1}{2}\operatorname{sinc}(n/2)$$

b) No, because
$$F(e^{j\mu}, e^{j0}) = rect(\mu/\pi) - rect(2\mu/\pi)$$
$$F(e^{j0}, e^{j\nu}) = rect(\nu/\pi) - rect(2\nu/\pi)$$
But
$$F(e^{j\mu}, e^{j\nu}) \not\propto F(e^{j\mu}, e^{j0})\, F(e^{j0}, e^{j\nu})$$
not proportional to

c)
$$rect(\mu/\pi, \nu/\pi) \overset{DSFT}{\Longleftrightarrow} \frac{1}{4}\operatorname{sinc}(m/2)\operatorname{sinc}(n/2)$$
$$rect(2\mu/\pi, 2\nu/\pi) \overset{DSFT}{\Longleftrightarrow} \frac{1}{16}\operatorname{sinc}(m/4)\operatorname{sinc}(n/4)$$
$$f(m,n) = \frac{1}{4}\operatorname{sinc}(m/2, n/2) - \frac{1}{16}\operatorname{sinc}(m/4)\operatorname{sinc}(n/4)$$

d) $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n) = F(e^{j0}, e^{j0})$

$= rect(0,0) - rect(0,0)$

$= 0$

Name: ______________________

Problem 5. (33 points)

Let $h(m,n) = (0.9)^{m+n} u(m)\, u(n)$

a) Calculate the 2-D Z-transform $H(z_1, z_2)$.

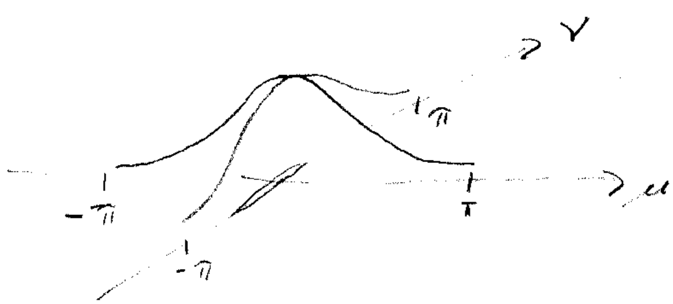
b) Calculate and sketch $\left|H(e^{j\mu}, e^{j\nu})\right|$ the magnitude of the 2-D DSFT. for $|\mu| < \pi$ and $|\nu| < \pi$.

c) Write a difference equation for the system with impulse response $h(m,n)$.

a) $$H(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (0.9)^{m+n} z_1^{-m} z_2^{-n}$$

$$= \frac{1}{1-0.9z_1^{-1}} \frac{1}{1-0.9z_2^{-1}}$$

b) $$|H(e^{j\mu}, e^{j\nu})| = \frac{1}{|1-0.9e^{j\mu}|} \frac{1}{|1-0.9e^{-j\nu}|}$$



c) $$(1-0.9z_1^{-1})(1-0.9z_2^{-1})\, Y(z) = X(z)$$

$$(1 - 0.9(z_1^{-1} + z_2^{-1}) + 0.81 z_1^{-1} z_2^{-1})\, Y(z) = X(z)$$

$$y(m,n) = x(m,n) + 0.9\,(y(m-1,n) + y(m,n-1))$$
$$- 0.81\, y(m-1, n-1)$$

Name: ______________________

Problem 6. (33 points)

Consider the function $f(x,y) = rect(x/2, y) + rect(x, y/2)$.

a) Sketch the function $f(x,y)$.
b) Let $g_\theta(t)$ be the projections of $f(x,y)$. Calculate $g_\theta(t)$ for $\theta = 0$ ~~and $\theta = \pi/2$.~~
c) Consider the shifted function $f(x - x_0, y - y_0)$. Calculate the $p_\theta(t)$, the projections of $f(x - \cancel{x_0}, y - \cancel{y_0})$, in terms of the function $g_\theta(t)$ (i.e. assume that you know $g_\theta(t)$ for all θ and t).
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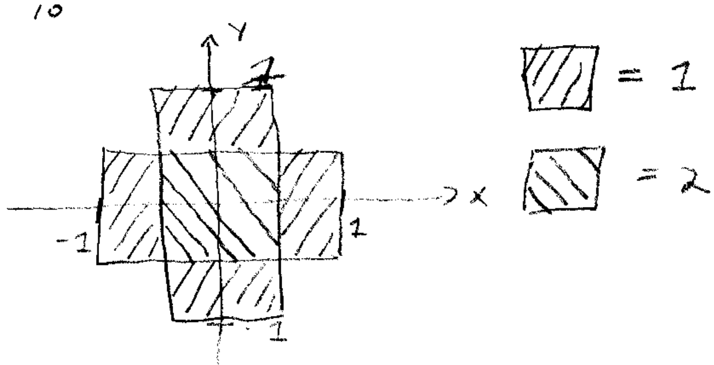
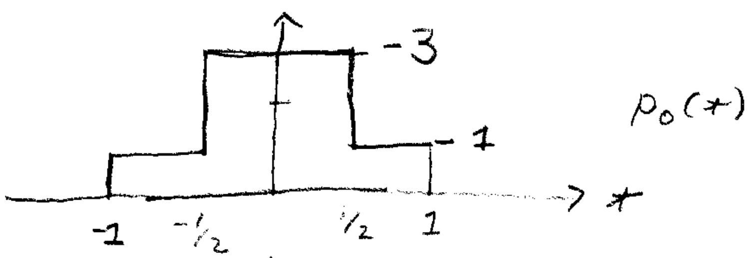
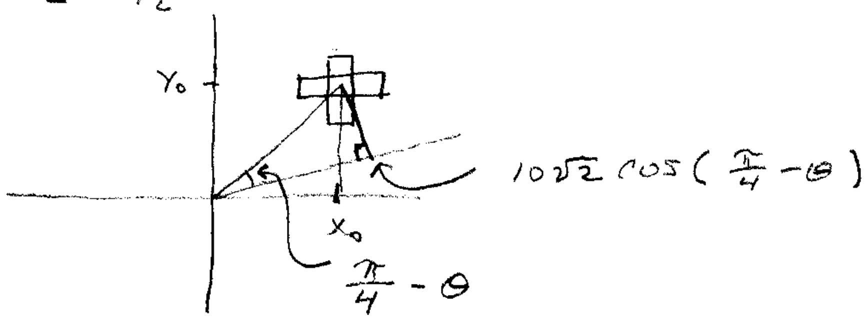
a)



b)



c)



$$p_\theta(t) = g_\theta\left(t - 10\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)\right)$$