Solutions Exam 2

1)

a) $H(e^{j\omega}) = \dfrac{1}{1-2z^{-1}} \cdot \dfrac{1}{1-\frac{1}{2}z^{-1}}$

Causal impulse response $\Rightarrow$ ROC $= \{ |z| > 2 \}$

$$H(e^{j\omega}) = \frac{A}{1-2z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{1}{1-\frac{1}{2}z^{-1}}\right|_{z=2} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$B = \left.\frac{1}{1-2z^{-1}}\right|_{z=\frac{1}{2}} = \frac{1}{1-4} = -\frac{1}{3}$$

$$H(e^{j\omega}) = \frac{4}{3}\frac{1}{1-2z^{-1}} - \frac{1}{3}\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$h(n) = \frac{4}{3}2^n u(n) - \frac{1}{3}(\tfrac{1}{2})^n u(n)$$

b) Stable impulse response $\Rightarrow$ ROC $= \{ \frac{1}{2} \le |z| \le 2 \}$

$$h(n) = \frac{-4}{3}2^n u(-1-n) - \frac{1}{3}(\tfrac{1}{2})^n u(n)$$

c) $h(n) = \dfrac{-4}{3} 2^n u(-1-n) + \dfrac{1}{3}(\tfrac{1}{2})^n u(-1-n)$

2) a)

$$X(e^{j\omega}) = \sum_{n=0}^{7} e^{-j\omega n}$$

$$= \frac{1-e^{-j\omega 8}}{1-e^{-j\omega}} = \frac{e^{-j\omega 4}}{e^{-j\omega/2}} \frac{(e^{+j\omega 4}-e^{-j\omega 4})}{(e^{+j\omega/2}-e^{-j\omega/2})}$$

$$= e^{-j\omega 3.5} \frac{\sin(\omega 4)}{\sin(\omega/2)}$$

b) $$X_{16}(k) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{16}} \quad k=0,\dots,15$$

$$X_{16}(k) = e^{-j\frac{2\pi k\, 3.5}{16}} \frac{\sin(2\pi k/4)}{\sin(2\pi k/32)}$$

c) $$X(k) = \sum_{n=0}^{15} \cos(2\pi n/N)$$

$$\frac{1}{2}\sum_{n=0}^{15}\left(e^{+j2\pi n/N}+e^{-j2\pi n/N}\right)e^{-j2\pi nk/16}$$

$$=\frac{1}{2}\sum_{n=0}^{15}e^{j2\pi n\left(\frac{1}{N}-\frac{k}{16}\right)}+\frac{1}{2}\sum_{n=0}^{15}e^{-j2\pi n\left(\frac{1}{N}+\frac{k}{16}\right)}$$

$$=\frac{1}{2}\,\frac{1-e^{j2\pi\left(\frac{1}{N}-\frac{k}{16}\right)16}}{1-e^{j2\pi\left(\frac{1}{N}-\frac{k}{16}\right)}}+\frac{1}{2}\,\frac{1-e^{j2\pi\left(\frac{1}{N}+\frac{k}{16}\right)16}}{1-e^{j2\pi\left(\frac{1}{N}-\frac{k}{16}\right)}}$$

$$=\frac{1}{2}\,\frac{e^{+j2\pi\left(\frac{1}{N}-\frac{k}{16}\right)8}}{e^{+j2\pi\left(\frac{1}{N}-\frac{k}{16}\right)/2}}\;\frac{\sin\left(2\pi\left(\frac{1}{N}-\frac{k}{16}\right)8\right)}{\sin\left(2\pi\left(\frac{1}{N}-\frac{k}{16}\right)/2\right)}$$

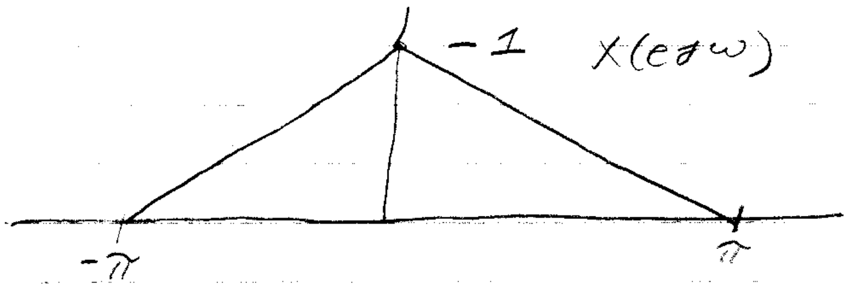
$$+\frac{1}{2}\,\frac{e^{-j2\pi\left(\frac{1}{N}+\frac{k}{16}\right)8}}{e^{-j2\pi\left(\frac{1}{N}+\frac{k}{16}\right)/2}}\;\frac{\sin\left(2\pi\left(\frac{1}{N}+\frac{k}{16}\right)8\right)}{\sin\left(2\pi\left(\frac{1}{N}+\frac{k}{16}\right)/2\right)}$$

$$=\frac{1}{2}\,e^{j2\pi\left(\frac{1}{N}-\frac{k}{16}\right)7.5}\;\frac{\sin\left(2\pi\left(\frac{1}{N}-\frac{k}{16}\right)8\right)}{\sin\left(2\pi\left(\frac{1}{N}-\frac{k}{16}\right)/2\right)}$$

$$+\frac{1}{2}\,e^{-j2\pi\left(\frac{1}{N}+\frac{k}{16}\right)7.5}\;\frac{\sin\left(2\pi\left(\frac{1}{N}+\frac{k}{16}\right)8\right)}{\sin\left(2\pi\left(\frac{1}{N}+\frac{k}{16}\right)/2\right)}$$

3)

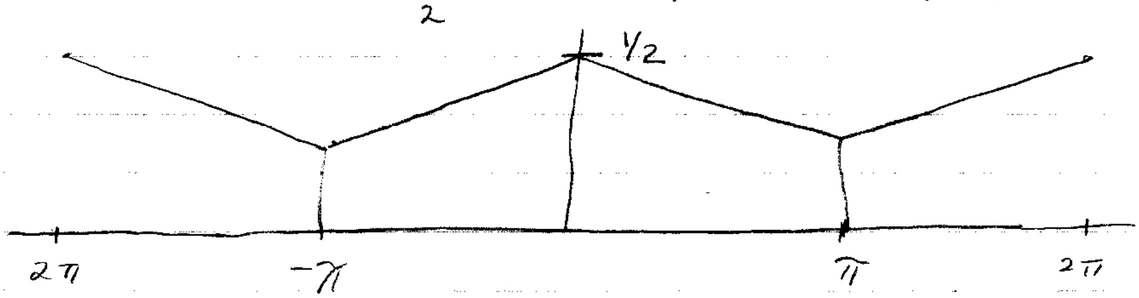
a)



b) $X(e^{j\omega})H(e^{j\omega}) = \Lambda(\omega/\pi)\,\mathrm{rect}(\omega/\pi) \quad |\omega|<\pi$

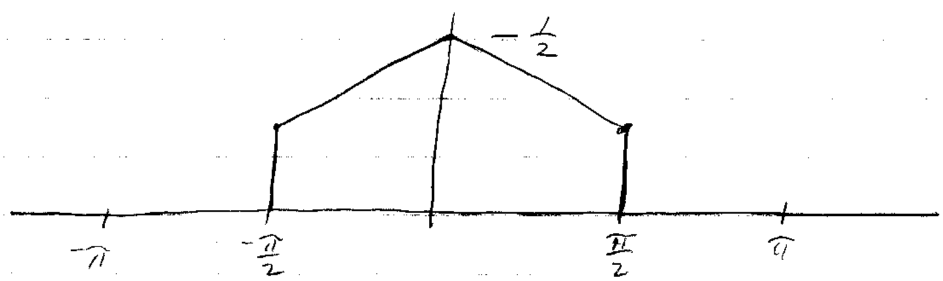
$$Z(e^{j\omega}) = \frac{1}{2}\,\Lambda(\omega/(2\pi))\,\mathrm{rec}(\omega/(2\pi)) \quad |\omega|<\pi$$

$$= \frac{1}{2}\,\Lambda(\omega/(2\pi)) \quad |\omega|<\pi$$



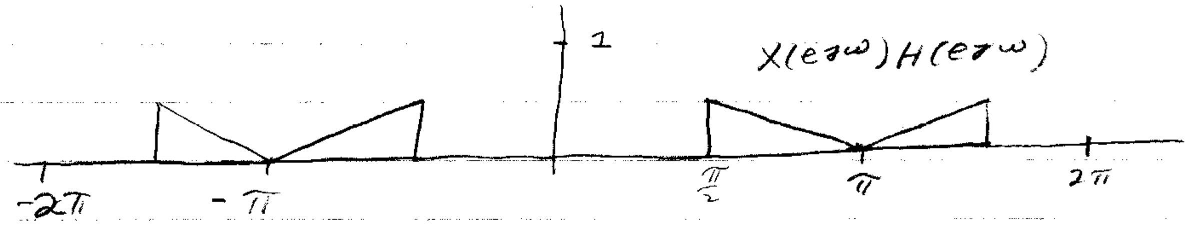
$$Y(e^{j\omega}) = Z(e^{j2\omega})\,\mathrm{rect}(\omega/\pi)$$

$$= \frac{1}{2}\,\Lambda(\omega/\pi)\,\mathrm{rect}(\omega/\pi)$$



c)

$$X(e^{j\omega})H(e^{j\omega}) = \Lambda(\omega/\pi)\left(1-\mathrm{rect}(\omega/\pi)\right) \quad |\omega| < \pi$$



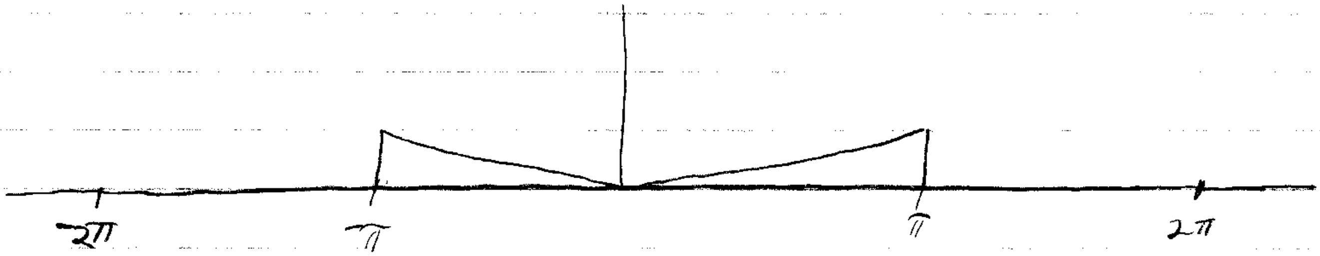
$$Z(e^{j\omega}) = \frac{1}{2}\sum_{k=0}^{1} X(e^{j(\omega-2\pi k)/2}) H(e^{j(\omega-2\pi k)/2})$$

$$= \frac{1}{2} X(e^{j\omega/2}) H(e^{j\omega/2}) \leftarrow 1^{st} \text{ term}$$

$$+\frac{1}{2} X(e^{j(\omega-2\pi)/2}) H(e^{j(\omega-2\pi)/2}) \leftarrow 2^{nd} \text{ term}$$

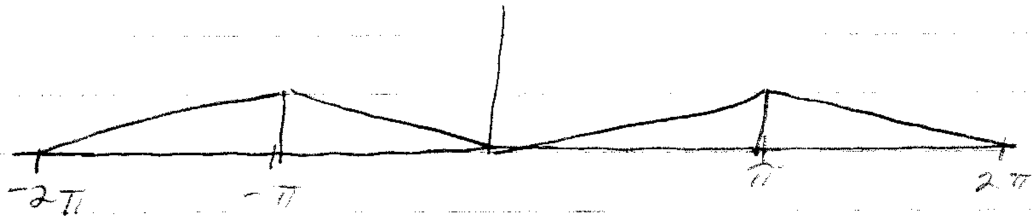
Notice that 1^{st} term $= 0$ for $|\omega| < \pi$

2^{nd} term looks like



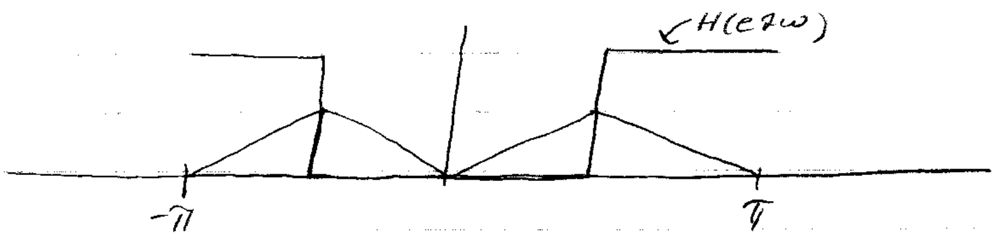
$$Z(e^{j\omega}) = \frac{1}{2}\Lambda((\omega+\pi)/\pi) + \Lambda((\omega-\pi)/\pi)$$

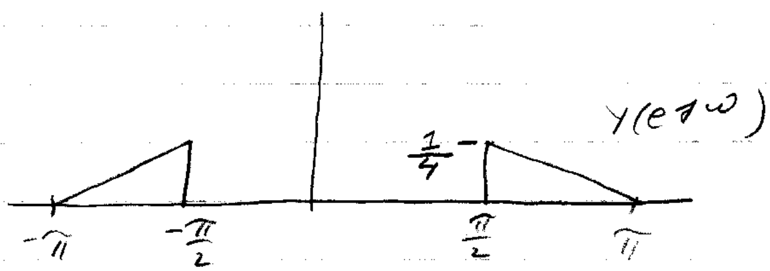
for $|\omega| < \pi$



$$Y(e^{j\omega}) = Z(e^{j2\omega})\,(1-\mathrm{rect}(\omega/\pi))$$

$|\omega| < \pi$





$$Y(e^{j\omega}) = \frac{1}{2}\Lambda(\omega/\pi)\,(1-\mathrm{rect}(\omega/\pi))$$