EE 438 Z-transform Example

Determine the frequency and impulse response of the following **causal** system.

$$y(n) = -\frac{1}{2}y(n-2) + x(n) + x(n-1)$$

Analysis:

$$Y(z) = \frac{1}{2}z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

$$Y(z)\left(1 + \frac{1}{2}z^{-2}\right) = X(z)\left(1 + z^{-1}\right)$$

$$H(z) = \frac{1 + z^{-1}}{1 + \frac{1}{2}z^{-2}}$$

$$= \frac{z(z+1)}{z^2 + \frac{1}{2}}$$

$$= \frac{z(z+1)}{\left(z + j\frac{1}{\sqrt{2}}\right)\left(z - j\frac{1}{\sqrt{2}}\right)}$$

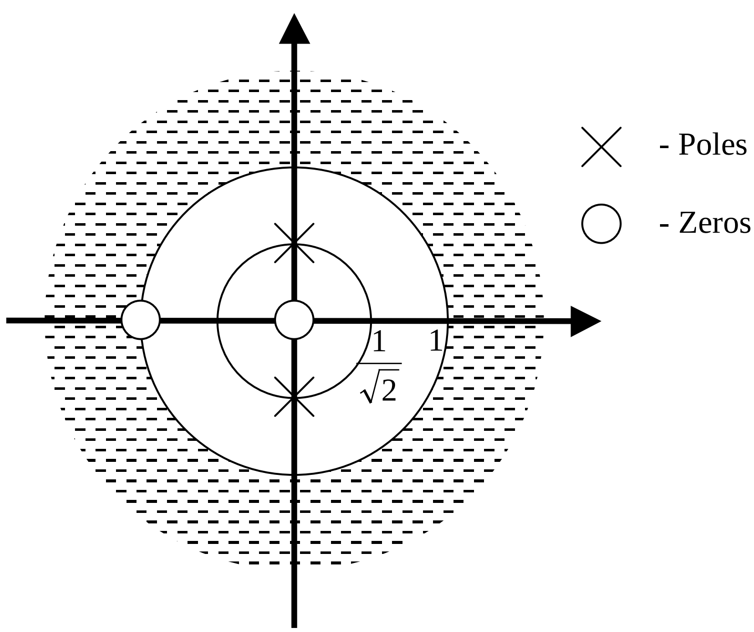
Then the region of convergence is:

Causal => $h(n)$ is right sided

⇨ ROC $= \{|z| > b\}$ where $b = \max_k |p_k| = \frac{1}{\sqrt{2}}$

⇨ ROC $= \{|z| > 1/\sqrt{2}\}$

Pole-zero diagram:



Compute impulse response

$$h(n) = Z^{-1}\{H(z)\}$$

Use partial fraction expansion (see appendix of Oppenheim, Willsky with Young)

$$\frac{H(z)}{z} = \frac{z+1}{\left(z + j\frac{1}{\sqrt{2}}\right)\left(z - j\frac{1}{\sqrt{2}}\right)} = \frac{A}{z + j\frac{1}{\sqrt{2}}} + \frac{B}{z - j\frac{1}{\sqrt{2}}}$$

$$\begin{aligned} A &= \left.\frac{z+1}{z - j/\sqrt{2}}\right|_{z=-j/\sqrt{2}} \\ &= \frac{1 - j/\sqrt{2}}{-j\frac{2}{\sqrt{2}}} \\ &= \frac{1}{2} + j\frac{\sqrt{2}}{2} \\ &= \frac{1}{2} + j\frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} B &= \left.\frac{z+1}{z + j/\sqrt{2}}\right|_{z=j/\sqrt{2}} \\ &= \frac{1 + j/\sqrt{2}}{j\frac{2}{\sqrt{2}}} \\ &= \frac{1}{2} - j\frac{\sqrt{2}}{2} \\ &= \frac{1}{2} - j\frac{1}{\sqrt{2}} \end{aligned}$$

Notice that $A = B^*$ because $-j/\sqrt{2}$ and $+j/\sqrt{2}$ are complex conjugate pole pairs.

$$\frac{H(z)}{z} = \frac{z\left(\frac{1}{2} + j\frac{1}{\sqrt{2}}\right)}{z + j\frac{1}{\sqrt{2}}} + \frac{z\left(\frac{1}{2} - j\frac{1}{\sqrt{2}}\right)}{z - j\frac{1}{\sqrt{2}}}$$

Since the ROC = $|z| > \frac{1}{\sqrt{2}}$

$$h(n) = \left(\frac{1}{2} + j\frac{1}{\sqrt{2}}\right)\left(-\frac{j}{\sqrt{2}}\right)^n u(n) + \left(\frac{1}{2} - j\frac{1}{\sqrt{2}}\right)\left(+\frac{j}{\sqrt{2}}\right)^n u(n)$$

Since $j = e^{j\pi/2}$ and $\left(\frac{1}{2} + j\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2} e^{j\tan^{-1}\sqrt{2}}$

$$\begin{aligned} h(n) &= \frac{\sqrt{3}}{2} e^{j\tan^{-1}\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n e^{-j\pi n/2} u(n) + \frac{\sqrt{3}}{2} e^{-j\tan^{-1}\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n e^{j\pi n/2} u(n) \\ &= \frac{\sqrt{3}}{2} \left(\frac{1}{\sqrt{2}}\right)^n 2\cos\left(\frac{\pi}{2} n - \tan^{-1}\sqrt{2}\right) u(n) \end{aligned}$$

$$h(n) = \sqrt{3}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{2} n - \tan^{-1}\sqrt{2}\right) u(n)$$