Filtering of random sequences

define $\mu_x(n) = E\{X_n\}$

$$r_{XX}(m,n) = E\{X_m X_n\}$$

Special case

A process is w.s.s. if

(1) $\mu_x(n) \equiv \mu_x$

(2) $r_{XX}(m,n) = r_{XX}(n-m, 0)$

$$\triangleq r_{XX}(n-m)$$

Note convention: 2nd argument − 1st argument

Consider $\quad r_{AB}[a,b] = r_{AB}[b-a]$

$$Y[n] = \sum_m h[n-m]\, X[m]$$

$$E\{Y_n\} = \sum_m h[n-m]\, E\{X_m\}$$

If X_n is w.s.s.

$$E\{Y_n\} = \sum_m h[n-m]\mu_x$$

$$= \sum_m h[m]\mu_x$$

$$= \mu_x \times \text{a constant}$$

To find r_{yy}, we first define cross-correlation

$$r_{xy}[m,n] = E\{X_m Y_n\}$$

$$= E\{X_m \sum_k h[n-k] X_k\}$$

$$= \sum_k h[n-k] E\{X_m X_k\}$$

$$= \sum_k h[n-k] r_{xx}[k-m]$$

let $l = k-m \Rightarrow k = m+l$

$$r_{XY}[m,n] = \sum_k h[n-(m+\ell)]\, r_{XX}(\ell)$$

$$= \sum_k h[n-m-\ell]\, r_{XX}(\ell)$$

$$\therefore\ r_{XY}[m,n] = r_{XX}[n-m]$$

↑ convolution sum

Then

$$r_{YY}[m,n] = E\{Y_m Y_n\}$$

$$= E\{Y_m \sum_k h[m-k] X_k\}$$

$$= \sum_k h[m-k]\, r_{XY}[m-k]$$

let $\ell = m-k \Rightarrow k = m-\ell$

$$= \sum_k h[n-(m-\ell)]\, r_{XY}(\ell)$$

$$= \sum_k h[n-m+\ell]\, r_{XY}(\ell)$$

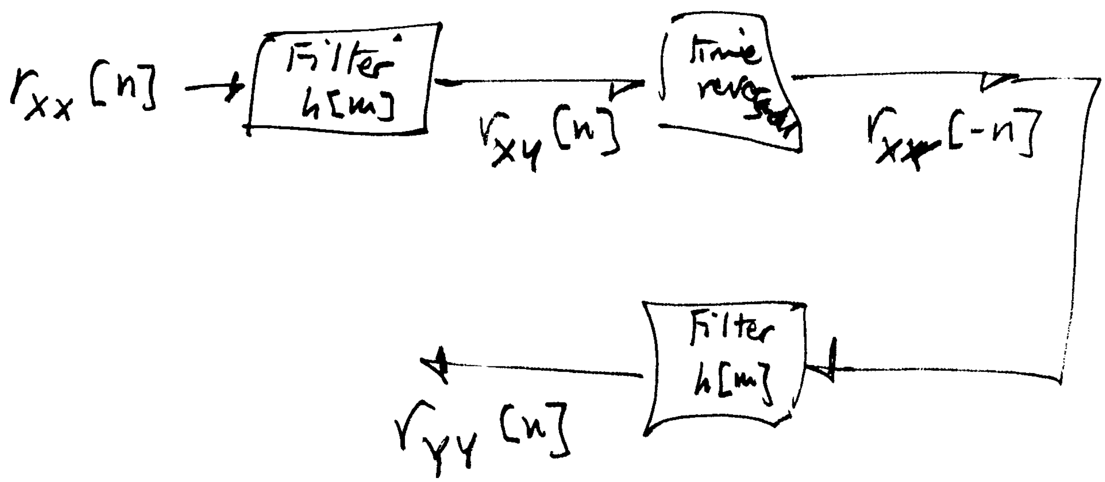
$$= \sum_k h[n-m-\ell]\, r_{XY}(-\ell)$$

So $r_{YY}[m,n] = r_{YY}[n-m]$

Summarizing:

If X is w.s.s., Y is also w.s.s.

and



~~Spectral Analysis~~

Estimating correlation

Assume X & Y are jtly w.s.s processes

Recall

$$r_{XY}[n] = E\{X[m]\,Y[m+n]\}$$

Suppose we form sum

$$z_M[n] = \frac{1}{M}\sum_{m=1}^{M} X[m]\,Y[m+n]$$

~~Then~~ $$\cancel{E\{z[n]\} = \frac{1}{M}\sum_{m=1}^{M} E\{X[m]\,Y[m+n]\}}$$

$$\cancel{= r_{XY}[n]}$$

~~Now lets suppose we compute $z[n]$ for $n = 1, \ldots, N$.~~
~~Lets look at this a little more closely:~~

<u>Interpretation of correlation</u>

Recall for X, Y two different random variables, we defined

<u>covariance</u>

$$\sigma_{xy}^2 = \frac{E\{(X-\mu_x)(Y-\mu_y)\}}{\cancel{\sigma_x \sigma_y}}$$

and <u>correlation coefficient</u>

$$\rho_{xy} = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y}$$

as measures of how <u>related</u> X & Y are, i.e. do they tend to vary <u>together</u> or do they <u>not</u>

Since $X[m]$ & $X[m+n]$ or $X[m]$ & $Y[m+n]$ are just two different pairs of r.v.s. we can see that

$$r_{XX}[n] = E\{X[m]X[m+n]\}$$

& $$r_{XY}[n] = E\{X[m]Y[m+n]\}$$

are closely related to covariance and correlation coefficient

(Suppose X & Y are zero mean)

So these have same interpretation i.e. how "related" are X_m & X_{m+n} or Y_n & Y_{n+m}?

Let's consider some examples

Suppose $X[n]$ is i.i.d with zero mean.

(Since mean of w.s.s. process is just a constant offset or shift we can let it $= 0$ without loss of generality.)

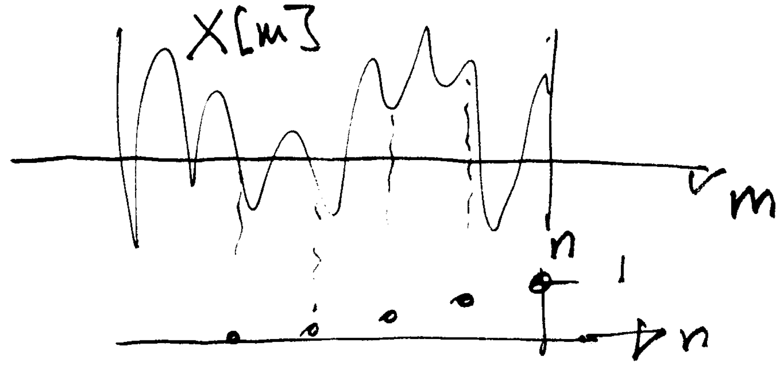
Then $r_{XX}[n] = E\{X[m]X[m+n]\}$

$$= \begin{cases} \sigma_X^2, & n = 0 \\ 0, & \text{else} \end{cases}$$

$$= \sigma_X^2\,\delta[n]$$

Now suppose

$$Y[n] = \sum_{m=0}^{7} 2^{-m}\,X[n-m]$$



This is obviously just a filtering operation with

$$h[n] = 2^{-n}\{u[n] - u[n-8]\}$$

We already know that

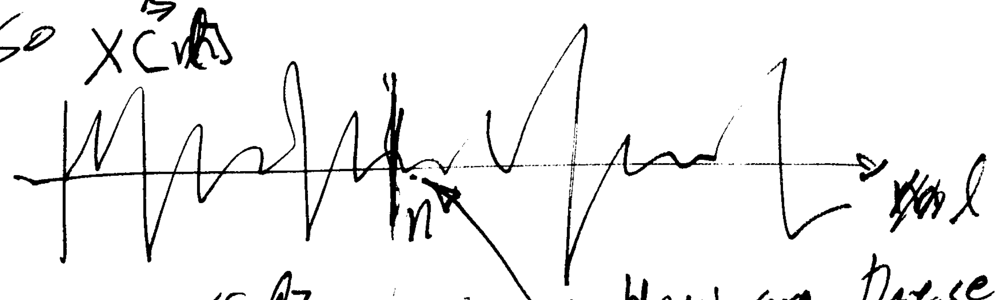
$$r_{xy}[n] = h[n] * r_{xx}[n]$$

note that $r_{xy}[n] = 0$ $n < 0$ & $n \geq 8$

$$= \sigma_x^2 \, h[n]$$

recall $\quad r_{xy}[n] = E\{X[m]\,Y[m+n]\}$

So



← How are these related?

do matlab example & show scatter plots for different values of n

What about the sequence $Y[n]$? Is it i.i.d?

Recall $r_{YY}[n] = h[n] * r_{XY}[-n]$

$$= h[n] * \sigma_X^2 h[-n]$$

$$= \sigma_X^2 \sum_m h[n-m]\, h[-m]$$

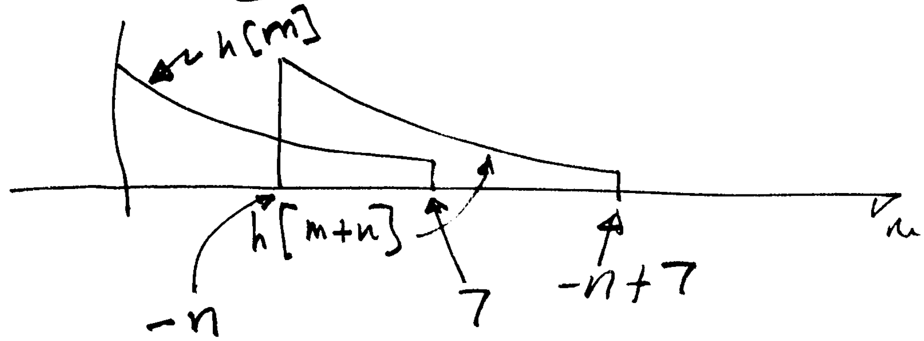
$$= \sigma_X^2 \sum_m 2^{-[n-m]}\{u[n-m] - u[n-m-8]\}\, 2^{-[-m]}\{u[-m] - u[-(m-8)]\}$$

$$= \sigma_X^2 \sum_{m=0}^{-7} h[n-m]\, h[-m]\, \{u[n-m] - u[n-m-8]\}$$

$$= \begin{cases} 0, & n < 0 \\ = & \end{cases}$$

$$r_{yy}[n] = \sigma_x^2 \sum_m h[n+m]\,h[m]$$

correlation sum



① Cases
$-n < -7$, $\quad r_{yy}[n] = 0$

② $-7 \le -n \le 0$

$$r_{yy}[n] = \sigma_x^2 \sum_{m=0}^{-n+7} 2^{-(n+m)}\, 2^{-m}$$

$$= \sigma_x^2\, 2^{-n} \sum_{m=0}^{-n+7} 2^{-2m} = \sigma_x^2\, 2^{-n}\, \frac{1 - 2^{-2(-n+8)}}{1 - 2^{-2}}$$

$$= \sigma_x^2\, \frac{4}{3}\left[2^{-n} - 2^{(n-16)}\right]$$

③ $0 < -n \le 7$

$$r_{yy}[n] = \sigma_x^2\, 2^{-n} \sum_{m=-n}^{7} 2^{-2m} \qquad l = m+n$$

$$= \sigma_x^2\, 2^{-n} \sum_{l=0}^{7+n} 2^{-2(l-n)}$$

(11)

$$= \sigma_x^2 \, 2^n \sum_{l=0}^{n+7} 2^{-2l} = \sigma_x^2 \, 2^n \, \frac{1 - 2^{-2(n+8)}}{1 - 2^{-2}}$$

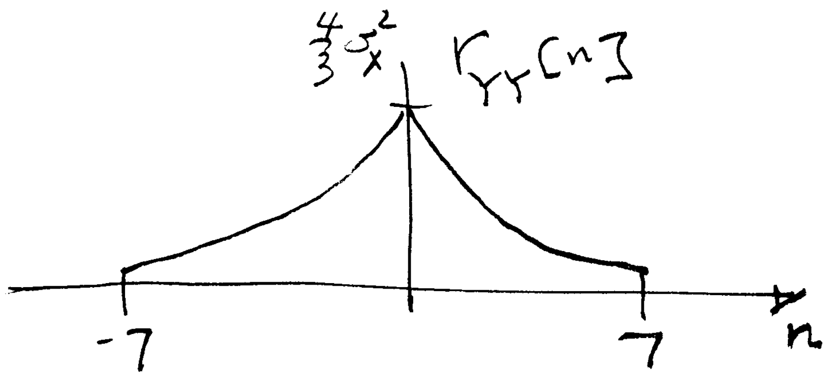
$$= \sigma_x^2 \, \frac{4}{3} \left[2^n - 2^{(-n-16)} \right]$$

(4) $7 \le -n$ $\qquad r_{yy}[n] = 0$

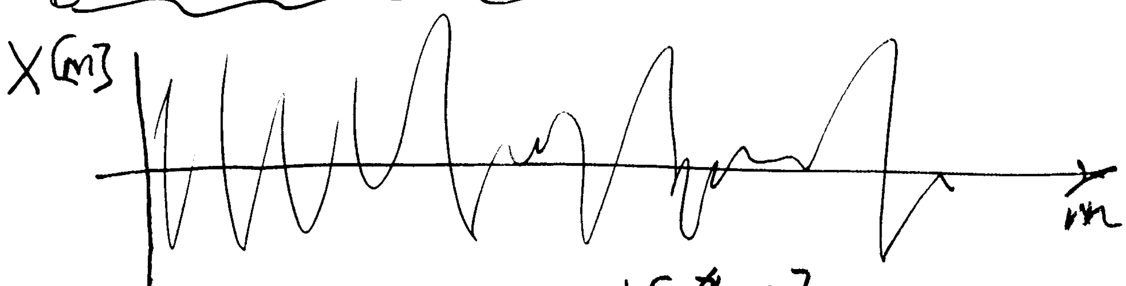
<u>Summarizing</u>

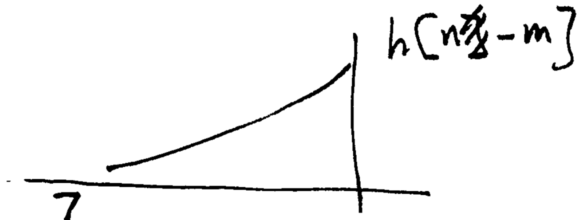
$$r_{yy}[n] = \begin{cases} 0, & n < -7 \\ \frac{4}{3}\sigma_x^2 \left[2^n - 2^{(-n-16)} \right], & -7 \le n \le 0 \\ \frac{4}{3}\sigma_x^2 \left[2^{-n} - 2^{(n-16)} \right], & 0 \le n \le 7 \\ 0, & n > 7 \end{cases}$$

$$\approx \begin{cases} 0, & n < -7 \\ \frac{4}{3}\sigma_x^2 \, 2^n, & -7 < n \le 0 \\ \frac{4}{3}\sigma_x^2 \, 2^{-n}, & 0 \le n \le 7 \\ 0, & n > 7 \end{cases}$$



Does this make sense?





$$Y[n] = \sum_{m=0}^{7} 2^{-m} X[n-m]$$