Two random variables

Described by joint density function

$$P\{a < X \le b \wedge c < Y \le d\} =$$

$$\int_a^b \int_c^d f_{XY}(x,y)\,dx\,dy$$

joint distribution function

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

Marginal densities

$$f_X(x) = \int f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int f_{XY}(x,y)\,dx$$

Expectation

$$E\{g(X,Y)\} = \iint g(x,y) f_{XY}(x,y)\,dx\,dy$$

Linearity

$$E\{ag(x,y) + bh(x,y)\} = a\,E\{g(x,y)\} + b\,E\{h(x,y)\}$$

Correlation

$$g(x,y) = xy$$

Covariance

$$g(x,y) = \frac{(x-\bar{X})(y-\bar{Y})}{\sigma_x \sigma_y}, \quad \overline{XY} = E\{g(x,y)\}$$

$$\sigma_{XY}^2 = E\{g(x,y)\}$$

$$= \frac{1}{\sigma_x \sigma_y}\left[E\{xy\} - E\{\bar{X}y\} - E\{x\bar{Y}\} + \bar{X}\bar{Y}\right]$$

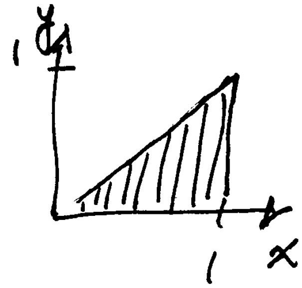
$$= \frac{1}{\sigma_x \sigma_y}\left\{\overline{XY} - \bar{X}\bar{Y}\right\}$$

Independence

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

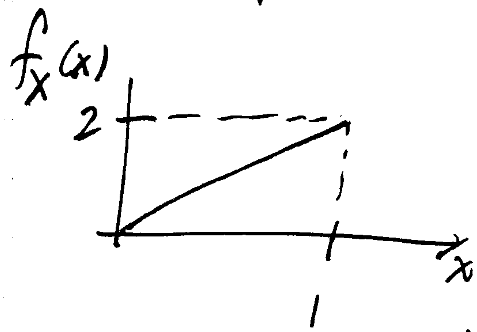
$$\Rightarrow \cancel{\sigma_{XY}^2 = 0} \quad \cancel{(XY} \quad \overline{XY} = \bar{X}\bar{Y} \ \& \ \sigma_{XY}^2 = 0$$

Example

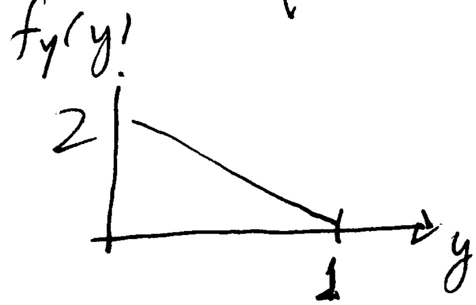


$$f_{XY}(x,y) = \begin{cases} 2, & 0 \le y < x \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_X(x) = \begin{cases} \int_0^x 2\, dy = 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$



$$f_Y(y) = \begin{cases} \int_y^1 2\, dx = 2(1-y), & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$



$$\overline{XY} = \int_{x=0}^{1} \int_{y=0}^{x} xy \, \cancel{f_{XY}(x,y)}\, 2 \, dx\, dy$$

$$= \int_{x=0}^{1} x \left\{ \int_{y=0}^{x} y \, \cancel{f_{XY}(x,y)}\, 2 \, dy \right\} dx$$

$$2 \frac{y^2}{2} \Big|_{y=0}^{x} = x^2$$

$$= \int_{x=0}^{1} x^3 dx = \frac{x^4}{4} \Big|_0^1 = \frac{1}{4}$$

$$\overline{X} = \int_0^1 x(2x)\, dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$\overline{Y} = \frac{1}{3}$$

$$\overline{X^2} = \int_0^1 x^2 (2x)\, dx = \frac{2}{4} x^4 \Big|_0^1 = \frac{1}{2}$$

$$\therefore \sigma_X^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$= \sigma_Y^2$$

$$\frac{\frac{9}{36} - \frac{8}{36}}{1/18} = \frac{1/36}{1/18} = \frac{1}{2}$$

$$\text{so } \sigma_{XY}^2 = \cancel{1/18} = \frac{\overline{XY} - \overline{X}\,\overline{Y}}{\sigma_X \sigma_Y} = \frac{1/4 - 2/9}{1/18} = \frac{1}{2}$$