Random Signals

Random Variables

denote by upper case X

completely characterized by **density function** $f_X(x)$

$$P\{ a < X \le b \} = \int_a^b f_X(x)\,dx$$

$$\Rightarrow f_X(x)\Delta x = P\{ a \le X \le x + \Delta x \}$$

distribution function

$$F_X(x) = P\{ X \le \cancel{b}\, x \}$$

$$= \int_{-\infty}^{x} f_X(\xi)\,d\xi$$

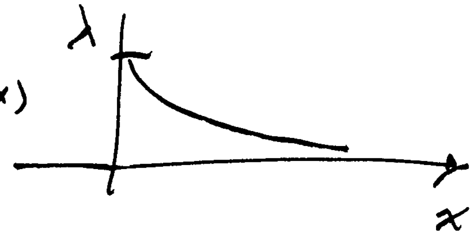
so $$\frac{dF_X(x)}{dx} = f_X(x)$$

Properties of density

1. $f_X(x) \ge 0$
2. $\int f_X(x)\,dx = 1$

<u>example</u>

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$



<u>check</u>

$$f_X(x) \geq 0$$

$$\int_0^\infty f_X(x)\,dx = \int_0^\infty \lambda e^{-\lambda x} dx = e^{-\lambda x}\Big|_\infty^0 = 1$$

What is prob. $X \leq 1$?

$$P\{X \leq 1\} = \int_0^1 \lambda e^{-\lambda x} dx$$

$$= e^{-\lambda x}\Big|_1^0 = 1 - e^{-\lambda}$$

<u>Expectation</u>

$$E\{g(X)\} = \int g(x) f_X(x)\,dx$$

where g is a function

Special cases:

Mean value

$$g(x) = x$$

$$\overline{X} = E\{X\} = \int x f_X(x)\,dx$$

2nd moment

$$g(x) = x^2$$

$$\overline{X^2} = E\{X^2\} = \int x^2 f_X(x)\,dx$$

Variance

$$g(x) = (x - \overline{x})^2$$

Expectation is a linear operator

$$E\{a g(X) + b h(X)\} = a E\{g(X)\} + b E\{h(X)\}$$

$$\therefore \sigma_X^2 = E\{(X - \overline{X})^2\} = E\{X^2 - 2X\overline{X} + \overline{X}^2\}$$

$$= E\{X^2\} - \overline{X}^2$$

example

$$E\{X\} = \int_0^{\infty} \lambda x e^{-\lambda x}\,dx$$

Integrate by parts:

$$\int u\,dv = uv - \int v\,du$$

$$u = \cancel{\lambda}x \qquad du = 1$$
$$dv = \lambda e^{-\lambda x} \qquad V = -e^{-\lambda x}$$

$$\int_0^\infty \lambda x\, e^{-\lambda x} dx = -x e^{-\lambda x}\Big|_0^\infty + \underbrace{\int_0^\infty e^{-\lambda x} dx}_{-\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty}$$

$$= \frac{1}{\lambda}$$

$$E\{x^2\} = \int_0^\infty \lambda x^2 e^{-\lambda x} dx$$

again, integrate by parts

$$u = x^2 \qquad du = 2x$$
$$dv = \lambda e^{-\lambda x} \qquad V = -e^{-\lambda x}$$

$$\int_0^\infty \lambda x^2 e^{-\lambda x} dx = -x^2 e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty x e^{-\lambda x} dx$$

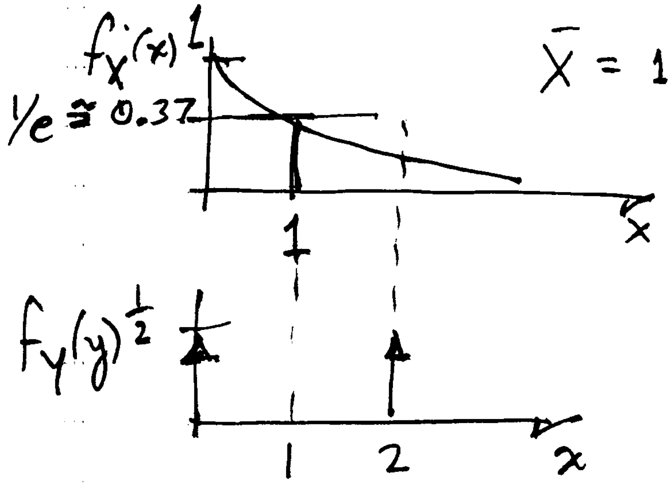
$$= \frac{2}{\lambda}$$

$$\sigma_X^2 = \frac{2}{\lambda} - \left(\frac{1}{\lambda}\right)^2$$

What does this mean?

Let $\lambda = 1$

$\bar{X} = 1$, $\sigma_X^2 = 1 \Rightarrow \sigma_X = 1$



X & Y have same mean & std. dev.

Transformations of a random variable:

$$Y = aX + b$$

$$\bar{Y} = a\bar{X} + b$$

$$\sigma_Y^2 = E\{(Y - \bar{Y})^2\}$$
$$= E\{[aX + b - (a\bar{X} + b)]^2\}$$
$$= a^2 E\{(X - \bar{X})^2\} = a^2 \sigma_X^2$$

- $$F_Y(y) = P\{Y \le y\}$$

$$= P\{aX + b \le y\} \qquad \text{assume } a > 0$$

$$= P\left\{X \le \frac{y-b}{a}\right\}$$

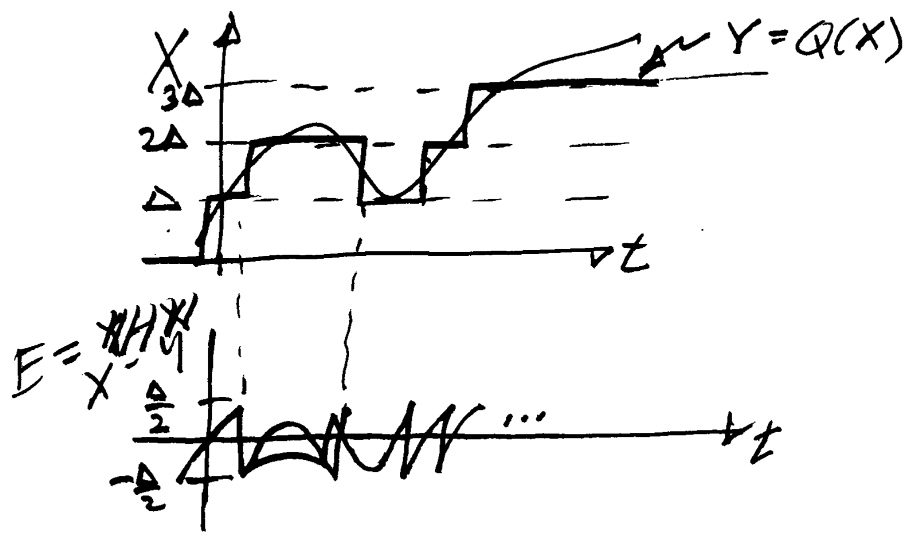
$$= F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

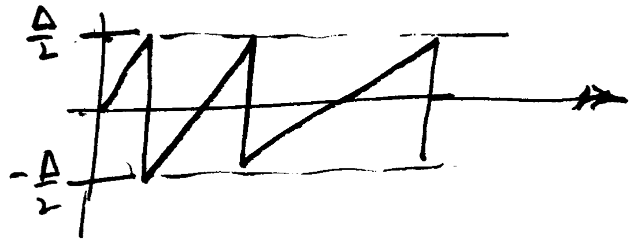
- in general

$$f_Y(y) = \left|\frac{1}{a}\right| f_X\left(\frac{y-b}{a}\right)$$

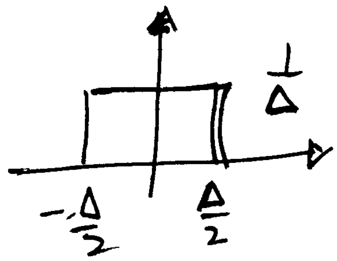
Application: Quantization



Approximation



What is density for ε?



$$E\{\varepsilon\} = 0$$

$$\sigma_\varepsilon^2 = \frac{2}{\Delta}\int_0^{\Delta/2} x^2\,dx = \frac{2}{\Delta}\left(\frac{\Delta}{2}\right)^3 \frac{1}{3}$$

$$= \frac{\Delta^2}{12}$$

SNR

Assume $N = 2^{B+1}$ levels uniformly distributed between $-\frac{(N-1)}{2}\Delta$ & $\frac{(N-1)}{2}\Delta$

Assume $N - 1 \approx N$

$$\sigma_x^2 = \frac{(N\Delta/2)^2}{12}$$

$$SNR = \cancel{N^2/4}\ 10 \log_{10}\left(\frac{N^2}{4}\right)$$

$$= 10 \log_{10}\left(\frac{2^{2B}}{4}\right)$$

$$= 20\left[B \log_{10}(2) - \log_{10}(2)\right]$$

$$= 6(B-1)$$