# Fourier Slice Theorem

$$P_\theta(\rho) = \int p_\theta(t) e^{-j2\pi\rho t} dt$$

$$= \int \left\{ \iint f(x,y)\, \delta(x\cos\theta + y\sin\theta - t)\, dx\, dy \right\} e^{-j2\pi\rho t} dt$$

$$= \iint f(x,y) \left\{ \int \delta(x\cos\theta + y\sin\theta - t)\, e^{-j2\pi\rho t} dt \right\} dx\, dy$$



$$= \iint f(x,y)\, e^{-j2\pi\rho[x\cos\theta + y\sin\theta]} dx\, dy$$

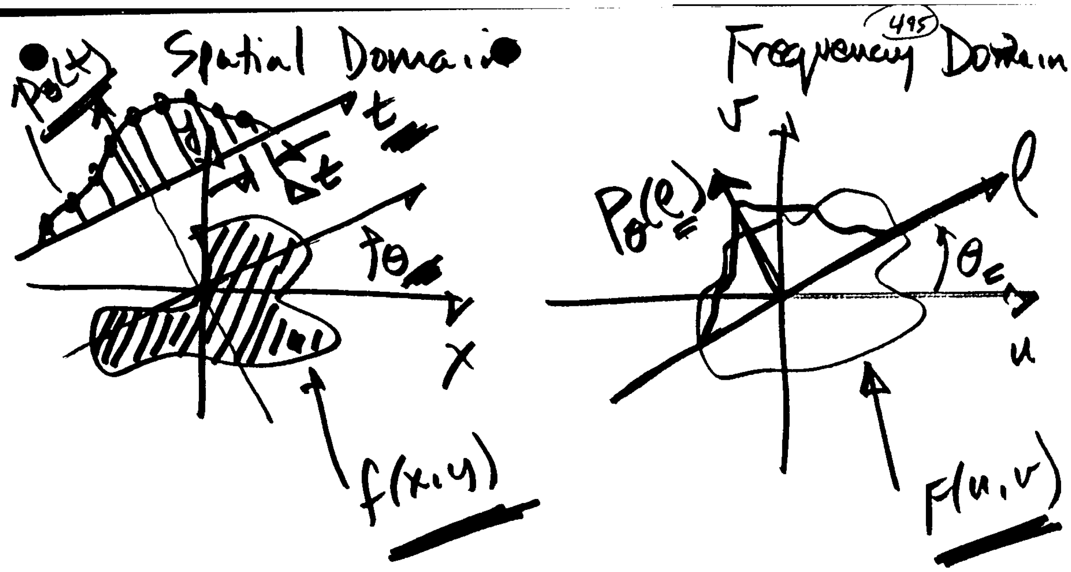
$u = \rho\cos\theta$
$v = \rho\sin\theta$

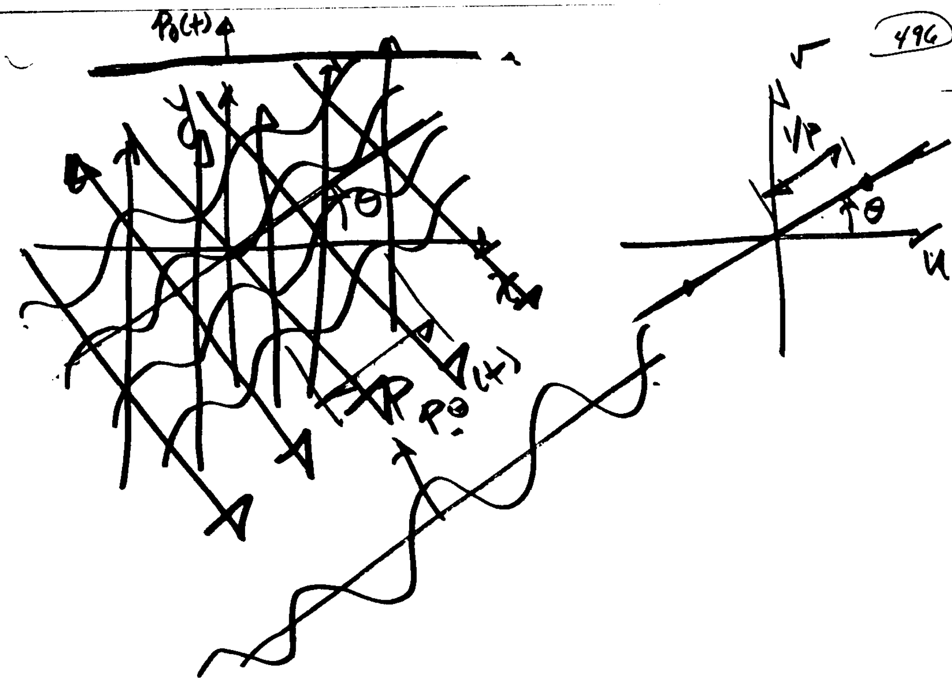
$$= F(u,v)$$

$$= F(\rho\cos\theta, \rho\sin\theta)$$

$$= \tilde{F}(\rho, \theta)$$

polar coordinate form of 2-D CSFT

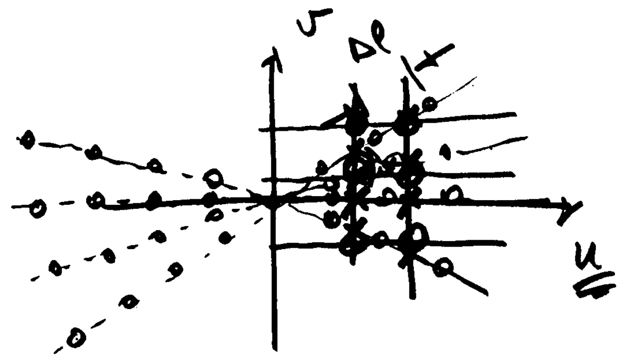



Line integrals cancel out all sinewaves except those orthogonal to direction of integration.

# Reconstruction Methods Based on Fourier Slice Theorem

## I. Direct Fourier Inversion




A. Interpolate from Polar to Cartesian grid



B. Take inverse 2D-DFT.

## II. Filter-Backprojection

Start with

polar coordinate form

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} \tilde{F}(\rho,\phi)\, e^{j2\pi\rho r\cos(\phi-\theta)}\,\rho\, d\rho\, d\phi$$

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$r\cos(\phi-\theta) = \cos\phi\,\underbrace{r\cos\theta}_{x} + \sin\phi\,\underbrace{r\sin\theta}_{y}$$

$$f(x,y) = \int_0^{2\pi} \int_0^\infty \tilde{F}(\rho,\phi) e^{j 2\pi\rho[x\cos\phi + y\sin\phi]} \rho\, d\rho\, d\phi$$ 499

$$= \int_0^{\pi} \int_0^\infty \tilde{F}(\rho,\phi) e^{j 2\pi\rho[x\cos\phi + y\sin\phi]} \frac{\rho\, d\rho\, d\phi}{|\rho|} \quad ①$$

$$+ \int_\pi^{2\pi} \int_0^\infty \tilde{F}(\rho,\phi) e^{j 2\pi\rho[x\cos\phi + y\sin\phi]} \rho\, d\rho\, d\phi \quad ②$$

let $\phi' = \phi - \pi \qquad \phi = \phi' + \pi$

$\cos(\phi'+\pi) = \cos(\phi')\cos(\pi) - \sin(\phi')\sin(\pi) = -\cos(\phi')$

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$$\tilde{F}(\rho, \phi'+\pi) = \int_0^{2\pi} \int_0^\infty \tilde{f}(r,\theta) e^{-j 2\pi(-\rho)\cos(\phi'-\theta)} r\, dr\, d\theta$$

$$= \tilde{F}(-\rho, \phi')$$

Integral ② becomes:

$$\int_0^{\pi} \int_0^\infty \tilde{F}(-\rho,\phi') e^{j 2\pi(-\rho)[x\cos\phi' + y\sin\phi']} \rho\, d\rho\, d\phi'$$

let $\rho' = -\rho$

$\sin(\phi'+\pi) = \sin(\phi')\cos(\pi) + \sin(\pi)\cos(\phi')$
$= -\sin(\phi')$

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(2) becomes

$$\int_0^{\pi}\int_0^{\infty} \underline{\tilde{F}(\rho, \phi'+\pi)}\, e^{j2\pi\rho[x\cos(\phi'+\pi) + y\sin(\phi'+\pi)]}\,\rho\, d\rho\, d\phi'$$

$$\cos(\phi'+\pi) = -\cos\phi'$$

$$\sin(\phi'+\pi) = -\sin\phi'$$

$$\tilde{F}(\rho, \phi'+\pi) = \int_0^{2\pi}\int_0^{\infty} \tilde{f}(r,\theta)\, e^{-j2\pi\rho\cos(\phi'+\pi-\theta)}\, r\, dr\, d\theta$$

(502)

Integral (2) becomes

$$\int_0^{\pi}\int_{-\infty}^{0} \tilde{F}(\rho', \phi')\, e^{j2\pi\rho'[x\cos\phi' + y\sin\phi']}\, [-\rho']\, d\rho'\, d\phi'$$

$$|\rho'|$$

$$\left[ f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} \underline{\tilde{F}(\rho,\phi)}\, e^{j2\pi\rho[\underline{x\cos\phi} + \underline{y\sin\phi}]}\, |\rho|\, d\rho\, d\phi \right]$$

let $t = x\cos\phi + y\sin\phi$

## Steps in Filtered Back projection

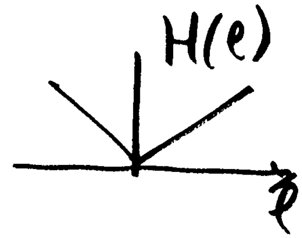
1. CFT of projection

$$P_\phi(\rho) = \hat{F}(\rho, \phi)$$

$$= \int p_\phi(t)\, e^{-j 2\pi \rho t}\, dt$$

2. Frequency domain filtering of projection data

$$Q_\phi(\rho) = \underline{|\rho|}\, P_\phi(\rho)$$






3. Take inverse CFT of filtered projection transform $Q_\phi(\rho)$

$$q_\phi(t) = \int_{-\infty}^{\infty} Q_\phi(\rho)\, e^{j 2\pi \rho t}\, d\rho$$

4. Back-projection

$$f(x, y) = \underline{\int_0^{\pi}} q_\phi(x\cos\phi + y\sin\phi)\, \underline{d\phi}$$

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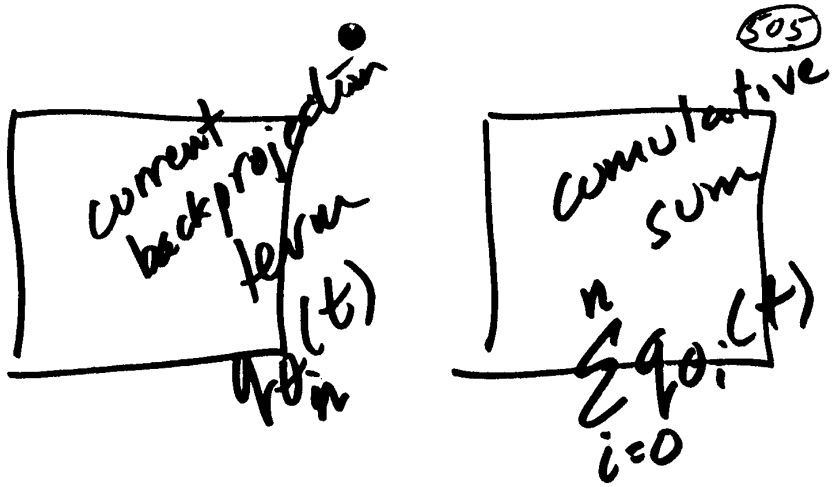



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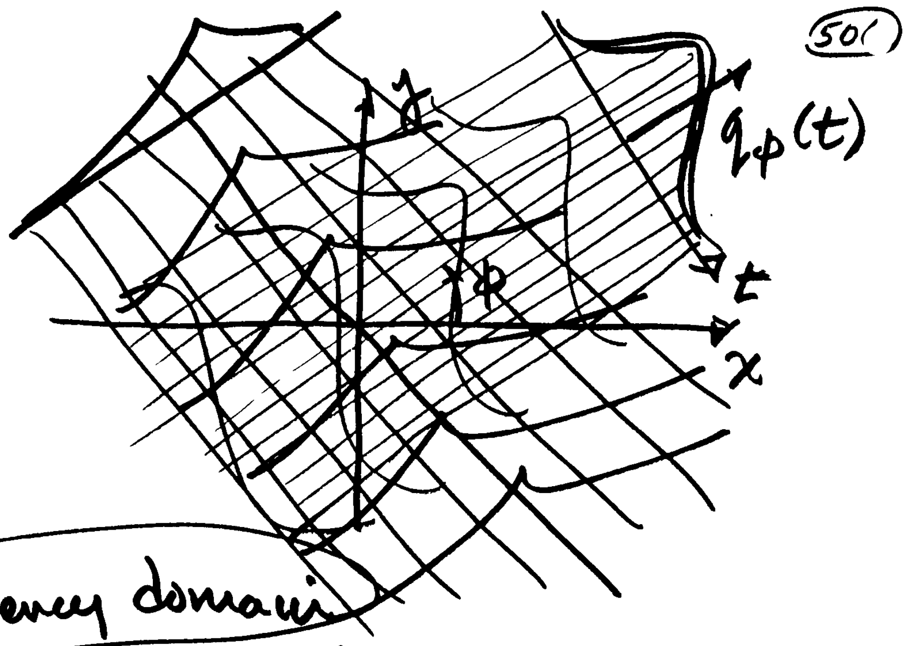


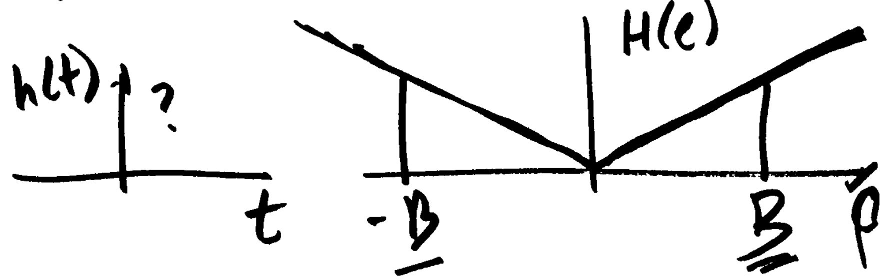

frequency domain
filtered projection — backprojection

## Convolution - Backprojection

(Spatial domain filtered projection-backprojection)

Replace steps ①, ②, ③ of earlier method by:

$$q_\phi(t) = \int h(t-\tau)\, \underline{p_\phi(\tau)}\, d\tau$$






psf does not exist for $H(\rho)$

Assume that $p_\phi(t)$ is essentially bandlimited to a highest spatial frequency B

$$P_\phi(\rho) = 0, \quad |\rho| > B$$

$$H_B(\rho) = |\rho|\, \mathrm{rect}\left(\frac{\rho}{2B}\right)$$

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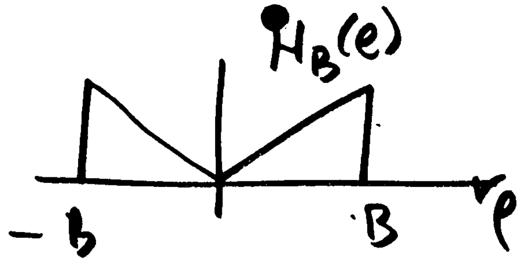


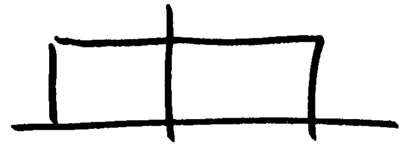

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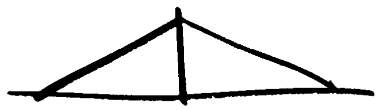

$$\Rightarrow \; h_B(t) = \underline{2B^2 \operatorname{sinc}(2Bt) - B^2 \operatorname{sinc}^2(Bt)}$$

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