Nonlinear Filtering

- Relatively new area
- Results are only beginning to appear in textbooks
- Theory is unlikely to ever be as compact as that for linear filters
- Much work is statistically based

Example

- Consider the following 1-D sequence:

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f[m]	2	2	2	14	2	2	2	2	4	5	6	7	8	8	8
g[m]															

- Moving average (linear) filter

$$g[m] = \frac{1}{3}(f[m-1] + f[m] + f[m+1])$$

- Assume boundary value is extended on left and on right

- Final result for linear filter

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f[m]	2	2	2	14	2	2	2	2	4	5	6	7	8	8	8
g[m]	2	2	6	6	6	2	2	$2\frac{2}{3}$	$3\frac{2}{3}$	5	6	7	$7\frac{2}{3}$	8	8

- Observations
  - smeared impulse (outlier) to width of filter
  - broadened transition region between two constant levels
  - these effects becoming increasingly pronounced as width of filter increases

- Observations (continued)

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f[m]	2	2	2	14	2	2	2	2	4	5	6	7	8	8	8
g[m]	2	2	6	6	6	2	2	$2\frac{2}{3}$	$3\frac{2}{3}$	5	6	7	$7\frac{2}{3}$	8	8

  - new signal values have been introduced
    $\Rightarrow$ must requantize
  - iterating filter will cause continued smearing of impulse and broadening of edges

- Median filter

$$g[m] = \text{median}(f[m-1], f[m], f[m+1])$$

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f[m]	2	2	2	14	2	2	2	2	4	5	6	7	8	8	8
g[m]															

- Final result for median filter

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f[m]	2	2	2	14	2	2	2	2	4	5	6	7	8	8	8
g[m]	2	2	2	2	2	2	2	2	4	5	6	7	8	8	8

- Observations
  - impulse is eliminated, independent of its amplitude
  - constant regions and the edge between them are all preserved
  - no new signal values occur
  - repeated filtering will have no further effect

Analysis of Standard 1-D Median Filter

Definitions

- *Length L input signal*: $f[m], m = 0,\ldots,L-1$

- *Length 2N + 1 median filter*:

$$g[m] = \text{median}(f[m-N],\ldots,f[m],\ldots,f[m+N])$$

- *Constant neighborhood*: a region of at least N + 1 consecutive points with same value

- *Edge*: a monotonically rising or falling set of points lying between two constant neighborhoods

- *Impulse*: any set of N or fewer points with values different from the surrounding regions which are identically-valued constant neighborhoods

Definitions (continued)

- *Oscillation*: a sequence of points that are not part of a constant neighborhood, edge, or impulse

- *Root*: a signal that is not modified by filtering

Properties of the Standard Median Filter

- Impulses are eliminated in one pass.
- A signal is a root if and only if it consists of only constant neighborhoods and edges.
- A root for filter size N is a root for all filters of size $M < N$.
- Repeated filtering will yield a root in at most $(L-1)/2$ passes (usually many fewer).

Properties (continued)

- The degree of smoothing increases with filter size N.

- The filter resolution (size of smallest detail passed) is $N+1$.

- To efficiently compute each output value, delete leftmost point in window and insert new point on right into sorted list ($0[\log_2(2N+1)]$ comparisons).

Example

Result of filtering oscillations with 3 point standard median filter

m		0	1	2	3	4	5	6	7	8	9	10
Initial Input		0	0	1	0	1	0	1	0	1	0	0
Outputs:	1st pass	0	0	0	1	0	1	0	1	0	0	0
	2nd pass	0	0	0	0	1	0	1	0	0	0	0
	3rd pass	0	0	0	0	0	1	0	0	0	0	0
	4th pass	0	0	0	0	0	0	0	0	0	0	0

Extensions to Standard Median Filter

Recursive Median Filter

$$g[m] = \text{median}(g[m-N], \ldots, g[m-1], f[m], f[m+1], \ldots, f[m+N])$$

- Properties
  - More smoothing for same size filter
  - Generates a root in one pass
  - Same root set as for standard median, but same input may go to different roots
  - Output not direction-invariant

Ranked Order Filters

Output n-th largest value in window

$$g[m] = n\text{–th largest value } (f[m-N], \ldots, f[m], \ldots, f[m+N])$$

- Special cases

n	filter type
1	minimum
$N+1$	median
$2N+1$	maximum

- Properties
  - tend to be peak or valley detectors
  - for $n \neq N+1$, only root signals are constant - valued
  - can also implement recursively.