SPATIAL FILTERING

Linear Filtering

- Each pixel value in the output image is a weighted sum of the pixels in the neighborhood of the corresponding pixel in the input image

- Sharpening
  - enhance edges and detail
  - boast higher frequency components

- Smoothing
  - remove noise

- Advantages of linear filters
  - rich theory of linear systems
  - ease of implementation

- Disadvantages of linear filters
  - may blur edges
  - outliers exert large influence on output

Notation

- f[m,n] - input image
- g[m,n] - output image
- h[m,n] - filter coefficients

Equations for linear filtering

- Simple weighted sum

$$g[m,n] = \sum_{k'} \sum_{\ell'} h'[k', \ell'] f[m+k', n+\ell']$$

Let $k = -k'$, $\ell = -\ell'$, $h[k, \ell] = h'[-k, -\ell]$

- Discrete convolution

$$g[m,n] = \sum_{k} \sum_{\ell} h[k, \ell] f[m-k,\ n-\ell]$$

- Alternate form for convolution

$$g[m,n] = \sum_{k} \sum_{\ell} h[m-k,\ n-\ell] f[k, \ell]$$

- Preferred form

$$g[m,n] = \sum_{k} \sum_{\ell} h[m-k,\ n-\ell] f[k,\ell]$$

- Filter is linear and shift-invariant

- Impulse response is h[m,n]

- To view impulse response as a function of (k, ℓ)

$$h[m-k,\ n-\ell] = h[-(k-m),\ -(\ell-n)]$$

Example 1

f[k,ℓ]

ℓ \ k	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	0	0	1	1	1	1
2	0	0	1	1	1	1
1	0	0	1	1	1	1
0	0	0	1	1	1	1

g[m,n]

n \ m	0	1	2	3	4	5
5						
4						
3						
2						
1						
0						

h[k,ℓ]

ℓ \ k	-1	0	1
1	1/16	1/8	1/16
0	1/8	1/4	1/8
-1	1/16	1/8	1/16

$$g[m,n] = \sum_{k} \sum_{\ell} h[-(k-m),\ -(\ell-n)]f[k,\ell]$$

Extending input image beyond the boundaries

- All zeros
- Extend boundary pixels outward
- Wrap around to opposite boundary

- Final Result

f[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	0	0	1	1	1	1
2	0	0	1	1	1	1
1	0	0	1	1	1	1
0	0	0	1	1	1	1

g[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	1/16	3/16	4/16	4/16	4/16
3	0	3/16	9/16	12/16	12/16	12/16
2	0	4/16	12/16	1	1	1
1	0	4/16	12/16	1	1	1
0	0	4/16	12/16	1	1	1

- Filter Characteristics
  - Smooths edges
  - Preserves input in areas that are constant over a region the size of the filter (DC preserving)

Spatial Domain Condition for DC Preserving Filter

Suppose $f[m,n] \equiv c$

$$g[m,n] = \sum_k \sum_\ell h[m-k,\ n-\ell]\ f[k,\ell]$$

$$= c \sum_k \sum_\ell h[m-k,\ n-\ell]$$

$$= c \sum_k \sum_\ell h[k,\ell]$$

$$g[m,n] \equiv c \iff \sum_k \sum_\ell h[k,\ell] = 1$$

Example 1 (earlier filter)

$h[k,\ell]$

ℓ		$k=-1$	$k=0$	$k=1$
	1	1/16	1/8	1/16
ℓ	0	1/8	1/4	1/8
	-1	1/16	1/8	1/16

Frequency Domain Analysis

Define 2-D extension of discrete time Fourier transform (DTFT):

Discrete Space Fourier Transform (DSFT)

- Forward transform

$$F(\mu,\nu) = \sum_{m}\sum_{n} f[m,n]\, e^{-j(m\mu + n\nu)}$$

- Inverse transform

$$f[m,n] = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(\mu,\nu)\, e^{j(m\mu + n\nu)}\, d\mu d\nu$$

- Spatial frequency variables

  μ - radians/sample in horizontal direction

  ν - radians/sample in vertical direction

  2π radians = 1 cycle

- Nyquist cutoff frequency

$$\pi \text{ radians/sample} = \frac{1}{2} \text{ cycle/sample}$$

Analysis of Discrete-Space, Linear, Shift-Invariant Filtering

$$g[m,n] = \sum_{k} \sum_{\ell} h[m-k,\ n-\ell]\ f[k,\ell]$$

$$G(\mu,\nu) = \sum_{m} \sum_{n} g[m,n]\ e^{-j(m\mu + n\nu)}$$

$$= \sum_{k} \sum_{\ell} \{\sum_{m} \sum_{n} h[m-k,\ n-\ell]\ e^{-j(m\mu + n\nu)}\}\ f[k,\ell]$$

$$G(\mu,\nu) = H(\mu,\nu) \sum_k \sum_\ell f[k,\ell]\, e^{-j(k\mu + \ell\nu)}$$
$$= H(\mu,\nu)\, F(\mu,\nu)$$

- Convolution theorem for 2-D discrete space signals

Example 1

$h[k,\ell]$

ℓ \ k	-1	0	1
1	1/16	1/8	1/16
0	1/8	1/4	1/8
-1	1/16	1/8	1/16

$h_1[\ell]$

ℓ	
1	1/4
0	1/2
-1	1/4

$h_1[k]$

k	-1	0	1
	1/4	1/2	1/4

$$h[k,\ell] = h_1[k]h_1[\ell]$$

$$H(\mu,\nu) = \sum_k \sum_\ell h_1(k)\, h_1(\ell)\, e^{-j(k\mu + \ell\nu)}$$

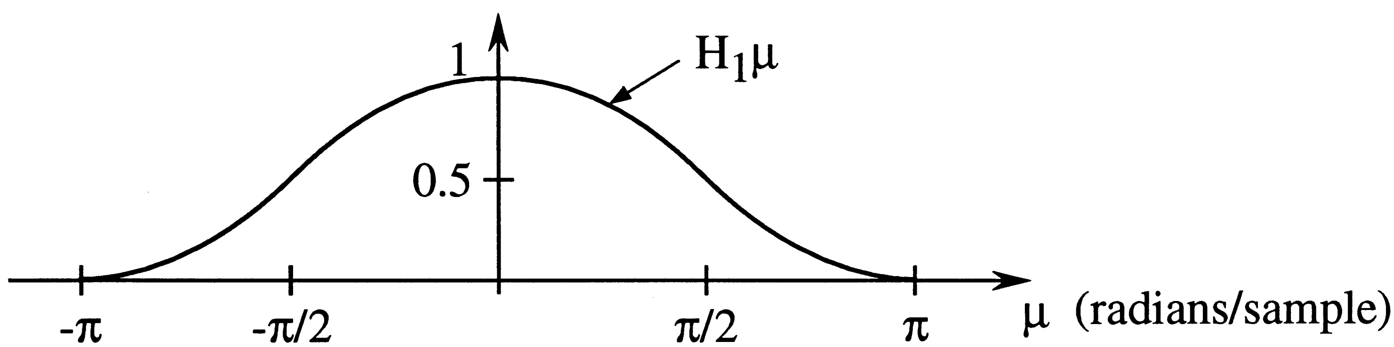
$$= [\sum_k h_1(k)\, e^{-jk\mu}]\ [\sum_\ell h_1(\ell)\, e^{-j\ell\nu}]$$

$$= H_1(\mu)\, H_1(\nu)$$

$$H_1(\mu) = \sum_k h_1(k)\, e^{-jk\mu}$$

$$= \frac{1}{4}\, e^{j\mu} + \frac{1}{2} + \frac{1}{4}\, e^{-j\mu}$$

$$= \frac{1}{2}[1 + \cos(\mu)]$$



Frequency Domain Condition for DC Preserving Filter

$$H(\mu,\nu) = \sum_{k} \sum_{\ell} h[k,\ell]\, e^{j(k\mu + \ell\nu)}$$

$$H(0,0) = \sum_{k} \sum_{\ell} h[k,\ell]$$

$$= 1$$

Example 2

f[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	0	0	1	1	1	1
2	0	0	1	1	1	1
1	0	0	1	1	1	1
0	0	0	1	1	1	1

g[m,n]

n \ m	0	1	2	3	4	5
5						
4						
3						
2						
1						
0						

h[k,ℓ]

ℓ \ k	-1	0	1
1	-1/9	-1/9	-1/9
0	-1/9	8/9	-1/9
-1	-1/9	-1/9	-1/9

- Final Result

f[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	0	0	1	1	1	1
2	0	0	1	1	1	1
1	0	0	1	1	1	1
0	0	0	1	1	1	1

g[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	-1/9	-2/9	-1/3	-1/3	-1/3
3	0	-2/9	5/9	1/3	1/3	1/3
2	0	-1/3	1/3	0	0	0
1	0	-1/3	1/3	0	0	0
0	0	-1/3	1/3	0	0	0

- Filter Characteristics
  - Large response at edges
  - No response where input is constant

$$\sum_k \sum_\ell h[k, \ell] = 0$$

Frequency Response

h[k,ℓ]

1	-1/9	-1/9	-1/9
0	-1/9	8/9	-1/9
-1	-1/9	-1/9	-1/9
	-1	0	1

=

0	0	0
0	1	0
0	0	0

–

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

1/3
1/3
1/3

1/3	1/3	1/3

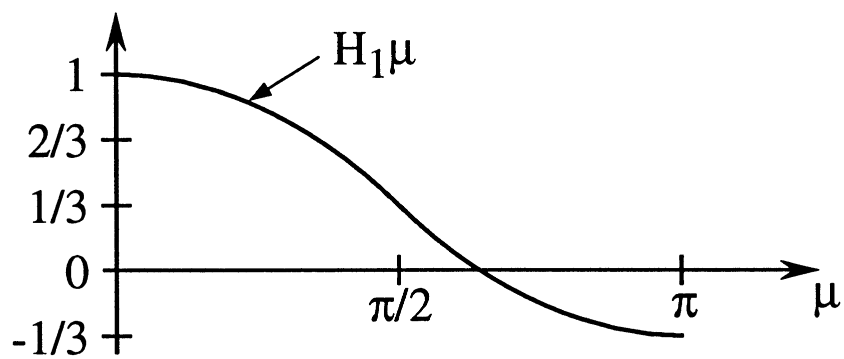
$$h[k,\ell] = \delta[k,\ell] - h_1[k]\, h_1[\ell]$$

$$H(\mu,\nu) = 1 - H_1(\mu)\, H_1(\nu)$$

$$H_1(\mu) = \sum_k h_1(k)\, e^{-jk\mu}$$

$$= \frac{1}{3}[e^{j\mu} + 1 + e^{-j\mu}]$$

$$= \frac{1}{3}[1 + 2\cos(\mu)]$$



Example 3 (Unsharp Mask)

$$g[m,n] = f[m,n] + \lambda\{f[m,n] - \langle f[m,n] \rangle\}$$

$\langle \bullet \rangle$ – spatial average over neighborhood of (m,n)-th pixel

$$h[k,\ell] = \delta[k,\ell] + \lambda\, h'[k,\ell]$$

$h'[k,\ell]$ - filter from Example 2

λ –non-negative parameter that controls amount of sharpening

- Filter coefficients

$h[k,\ell]$

ℓ \ k	-1	0	1
1	$-\lambda/9$	$-\lambda/9$	$-\lambda/9$
0	$-\lambda/9$	$1+\frac{8\lambda}{9}$	$-\lambda/9$
-1	$-\lambda/9$	$-\lambda/9$	$-\lambda/9$

- Filter is DC Preserving

- Effect on Image

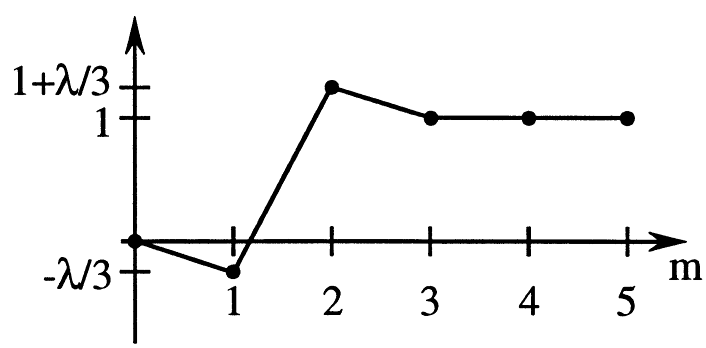
f[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	0	0	1	1	1	1
2	0	0	1	1	1	1
1	0	0	1	1	1	1
0	0	0	1	1	1	1

f[m,n]

n \ m	0	1	2	3	4	5
5	0	0	0	0	0	0
4	0	$-\lambda/9$	$-2\lambda/9$	$-\lambda/3$	$-\lambda/3$	$-\lambda/3$
3	0	$-2\lambda/9$	$1+\frac{5\lambda}{9}$	$1+\frac{\lambda}{3}$	$1+\frac{\lambda}{3}$	$1+\frac{\lambda}{3}$
2	0	$-\lambda/3$	$1+\frac{\lambda}{3}$	1	1	1
1	0	$-\lambda/3$	$1+\frac{\lambda}{3}$	1	1	1
0	0	$-\lambda/3$	$1+\frac{\lambda}{3}$	1	1	1

- Edge Profile



Frequency Response of Unsharp Mask

$$h[k,\ell] = \delta[k,\ell] + \lambda\, h'[k,\ell]$$

$$\begin{aligned} H(\mu,\nu) &= 1 + \lambda\, H'(\mu,\nu) \\ &= 1 + \lambda[1 - H_1(\mu)\, H_1(\nu)] \\ &= (1 + \lambda) - \lambda\, H_1(\mu)\, H_1(\nu) \end{aligned}$$