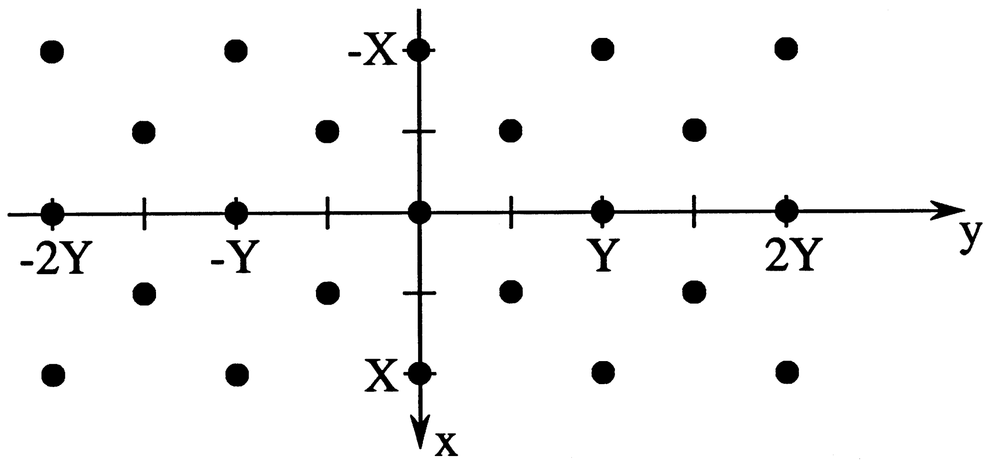
2.2.4 SAMPLING ON NON-RECTANGULAR LATTICES

Consider lattice structure of following form



Represent as two interlaced rectangular lattices

$$q(x,y) = \text{rep}_{X,Y}\left[\delta(x,y) + \delta(x - X/2, y - Y/2)\right]$$

$$Q(u,v) = \frac{1}{XY}\,\text{comb}_{1/X\,1/Y}\left[1 + e^{j2\pi[uX/2 + vY/2]}\right]$$

$$= \frac{1}{XY}\sum_k\sum_\ell\left[1 + e^{-j2\pi[(\frac{k}{X})X/2 + (\frac{\ell}{Y})Y/2]}\right] \times \delta(u - k/X, v - \ell/Y)$$

$$= \frac{1}{XY}\sum_k\sum_\ell\left\{1 + e^{-j\pi(k+\ell)}\right\}\delta(u - k/X, v - \ell/Y)$$

$$g_s(x,y) = q(x,y)g(x,y)$$

$$G_s(u,v) = Q(u,v) ** G(u,v)$$

$$= \frac{1}{XY} \sum_k \sum_\ell \left\{1 + e^{-j\pi(k+\ell)}\right\} G(u - k/X, v - \ell/Y)$$

Note that

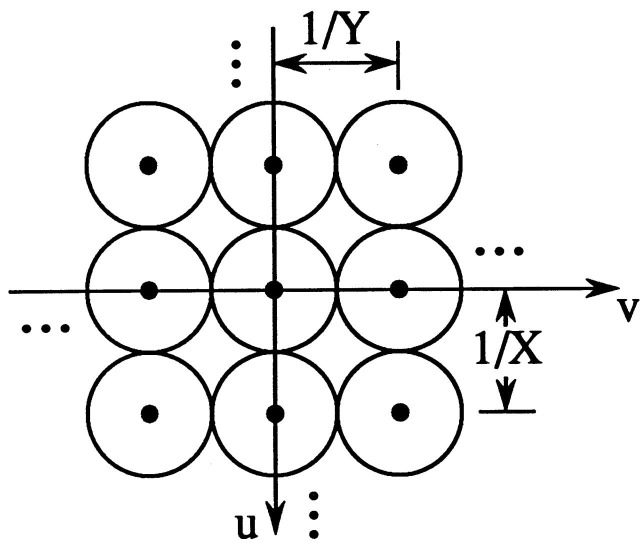
$$\left\{1 + e^{-j\pi(k+\ell)}\right\} = \begin{cases} 2, & k+\ell \quad \text{even} \\ 0, & k+\ell \quad \text{odd} \end{cases}$$

$\therefore$ Reciprocal lattice has same structure as spatial lattice.

Sampling of Circularly Band-limited Signals

$$G(u,v) = 0, \quad (u/U)^2 + (v/U)^2 > 1$$

Rectangular Lattice

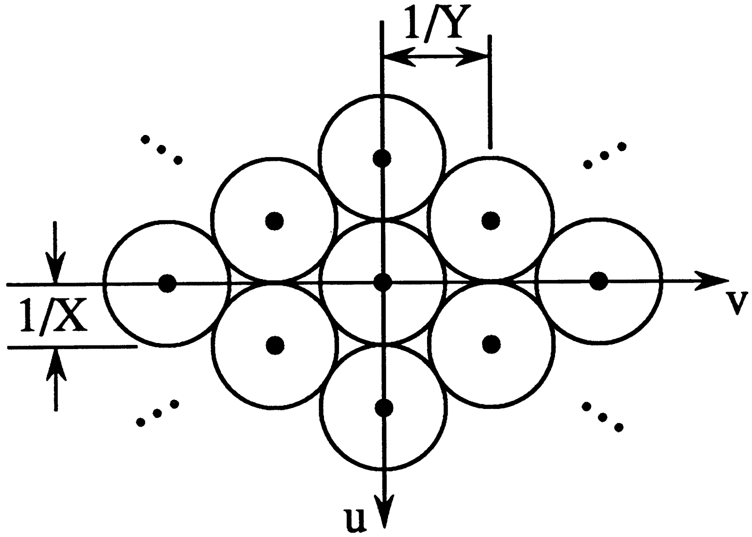


$$1/X = 1/Y = 2U$$

Sampling density

$$d_R = \frac{1}{XY} = 4U^2 \text{ samples/unit area}$$

Non-rectangular (hexagonal) lattice



$$1/X = U \quad 1/Y = \sqrt{4U^2 - U^2} = \sqrt{3}\,U$$

Sampling density

$$d_H = \frac{2}{XY} = 2\sqrt{3}\,U^2 \quad \text{samples/unit area}$$

$$\frac{d_H}{d_R} = \frac{2\sqrt{3}\,U^2}{4U^2} = \frac{\sqrt{3}}{2} = 0.866$$

$\Rightarrow$ 13.4% savings

When $X = 1/U$ and $Y = 1/(\sqrt{3}\,U)$, each lattice point is equidistant from its 6 nearest neighbors, so lattice is hexagonal.

Linear Algebra Formalism

Spatial coordinates

$$\vec{x} = (x, y)^T$$

Sampling matrix

$$X = \left[\vec{x}_1, \vec{x}_2\right]$$

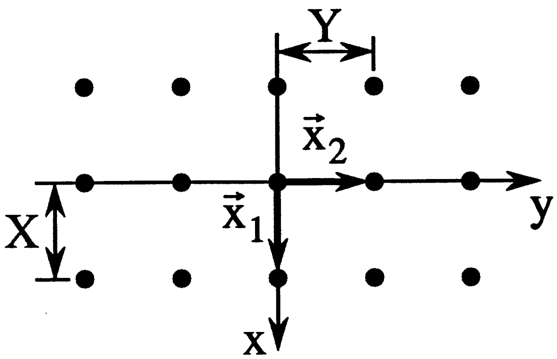
where $\vec{x}_1$ and $\vec{x}_2$ are basis vector which define the sampling lattice

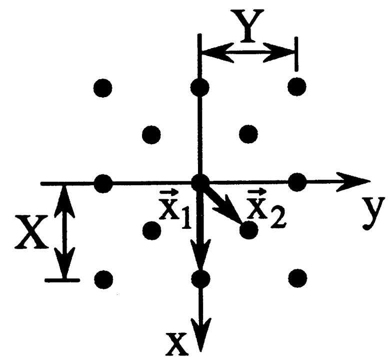
Integer vector

$$\vec{n} = (m, n)^T$$

$$q(\vec{x}) = \sum_{\vec{n}} \delta(\vec{x} - X\vec{n})$$

Examples





$$X_R = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

$$X_H = \begin{bmatrix} X & X/2 \\ 0 & Y/2 \end{bmatrix}$$

Sampling density

$$d = \left|\det[\mathbf{X}]\right|^{-1} \triangleq \left|\mathbf{X}\right|^{-1}$$

Examples

$$d_R = \left|\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right|^{-1} = \frac{1}{XY} \qquad d_H = \left|\begin{bmatrix} X & X/2 \\ 0 & Y/2 \end{bmatrix}\right|^{-1} = \frac{2}{XY}$$

Fourier analysis

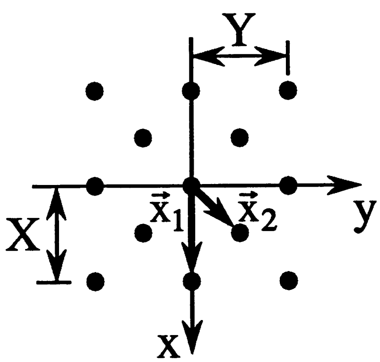
Reciprocal lattice $\mathbf{U}$ satisfies

$$\mathbf{U}^{\mathrm{T}}\mathbf{X} = \mathbf{I}$$
$$\Rightarrow \quad \mathbf{U} = (\mathbf{X}^{\mathrm{T}})^{-1}$$

Examples

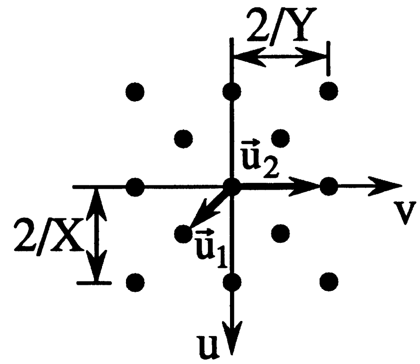
$$\mathbf{U}_{\mathrm{R}} = \begin{bmatrix} 1/\mathrm{X} & 0 \\ 0 & 1/\mathrm{Y} \end{bmatrix} \qquad \mathbf{U}_{\mathrm{H}} = \begin{bmatrix} 1/\mathrm{X} & 0 \\ -1/\mathrm{Y} & 2/\mathrm{Y} \end{bmatrix}$$

Spatial Domain



$$X_H = \begin{bmatrix} X & X/2 \\ 0 & Y/2 \end{bmatrix}$$

Frequency Domain



$$U_H = \begin{bmatrix} 1/X & 0 \\ -1/Y & 2/Y \end{bmatrix}$$

Frequency coordinates

$$\vec{u} = (u, v)^T$$

Fourier transform of sampling function

$$Q(\vec{u}) = \left|\mathbf{X}\right|^{-1} \sum_{\vec{k}} \delta(\vec{u} - \mathbf{U}\vec{k})$$

Fourier transform of sampled image

$$G_S(\vec{u}) = Q(\vec{u}) ** G(\vec{u})$$

$$= \left|\mathbf{X}\right|^{-1} \sum_{\vec{k}} G(\vec{u} - \mathbf{U}\vec{k})$$