2.1.2 2-D CONTINUOUS-SPACE FOURIER TRANSFORM (CSFT)

Forward transform

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux + vy)} dxdy$$

Inverse transform

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux + vy)} dudv$$

Hermitian Symmetry for Real Signals

Let

$$F(u,v) = A(u,v)\, e^{j\theta(u,v)}$$

If f(x,y) is real,

$$\begin{aligned} & F(u,v) = F^*(-u,-v) \\ \Rightarrow\ & A(u,v) = A(-u,-v) \quad \text{even symmetry} \\ & \theta(u,v) = -\,\theta(-u,-v) \quad \text{odd symmetry} \end{aligned}$$

In this case, the inverse transform may be written as

$$f(x,y) = 2\int_0^{\infty}\int_{-\infty}^{\infty} A(u,v)\cos[2\pi(ux + vy) + \theta(u,v)]\,du\,dv$$

2-D Transform Relations

1. linearity

$$a_1 f_1(x,y) + a_2 f_2(x,y) \overset{\text{CSFT}}{\leftrightarrow} a_1 F_1(u,v) + a_2 F_2(u,v)$$

2. scaling and shifting

$$f\left(\frac{x-x_0}{a}, \frac{y-y_0}{b}\right) \overset{\text{CSFT}}{\leftrightarrow} |ab| \; F(au, bv) \; e^{-j2\pi(ux_0+vy_0)}$$

3. modulation

$$f(x,y) \; e^{j2\pi(u_0x+v_0y)} \overset{\text{CSFT}}{\leftrightarrow} F(u - u_0, \; v - v_0)$$

4. reciprocity

$$F(x,y) \overset{\text{CSFT}}{\leftrightarrow} f(-u,-v)$$

5. Parseval's relation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)|^2 dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u,v)|^2 dudv$$

6. Initial value

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = F(0,0)$$

Separability

A function f(x,y) is *separable* if it factors as:

$$f(x,y) = g(x)h(y)$$

Some important separable functions:

$$\begin{aligned} \text{rect}(x,y) &= \text{rect}(x)\ \text{rect}(y) \\ \text{sinc}(x,y) &= \text{sinc}(x)\ \text{sinc}(y) \\ \delta(x,y) &= \delta(x)\ \delta(y) \end{aligned}$$

Transform Relation for Separable Functions

7. Let

$$g(x) \overset{\text{1-D CSFT}}{\leftrightarrow} G(u)$$

$$h(y) \overset{\text{1-D CSFT}}{\leftrightarrow} H(v)$$

then

$$g(x)\,h(y) \overset{\text{2-D CSFT}}{\leftrightarrow} G(u)\,H(v)$$

Important Transform Pairs

1. $\text{rect}(x,y) \overset{\text{CSFT}}{\leftrightarrow} \text{sinc}(u,v)$

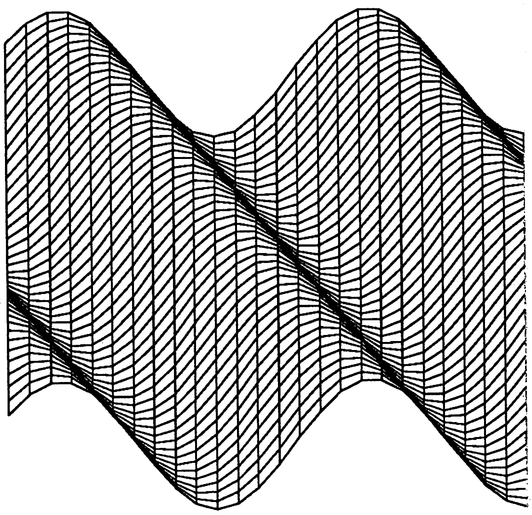
2. $\text{circ}(x,y) \overset{\text{CSFT}}{\leftrightarrow} \text{jinc}(u,v)$

3. $\delta(x,y) \overset{\text{CSFT}}{\leftrightarrow} 1$ (by sifting property)

4. $1 \overset{\text{CSFT}}{\leftrightarrow} \delta(u,v)$ (by reciprocity)

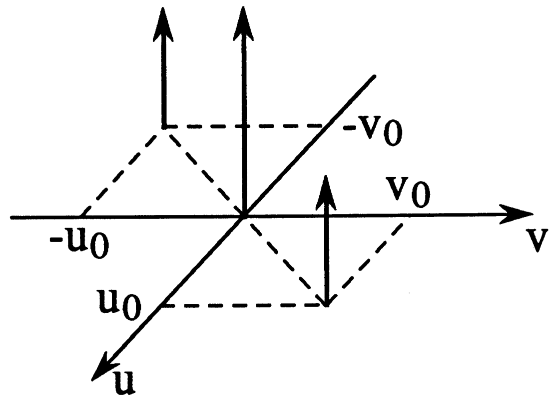
5. $e^{j2\pi[u_0x+v_0y]} \overset{\text{CSFT}}{\leftrightarrow} \delta(u-u_0,\ v-v_0)$
(by modulation property)

6. $\cos[2\pi(u_0x+v_0y)] \overset{\text{CSFT}}{\leftrightarrow} \frac{1}{2}[\delta(u-u_0, v-v_0) + \delta(u+u_0, v+v_0)]$

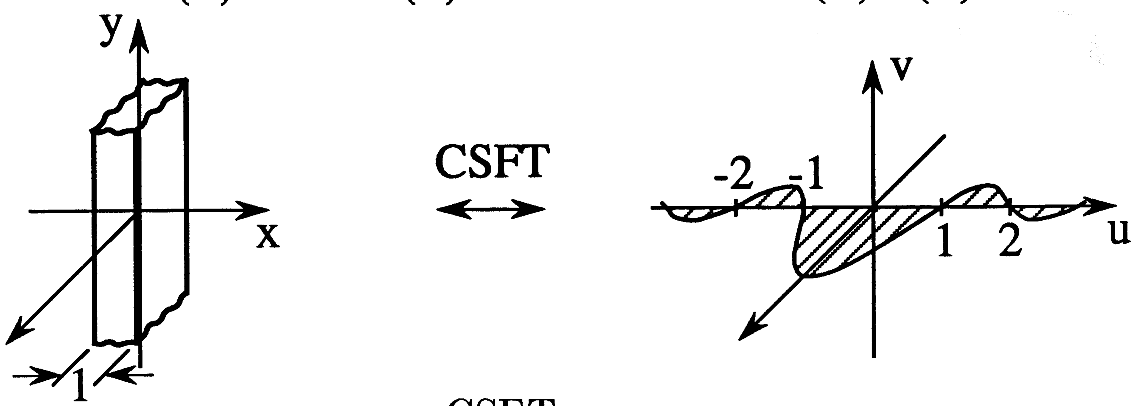


CSFT

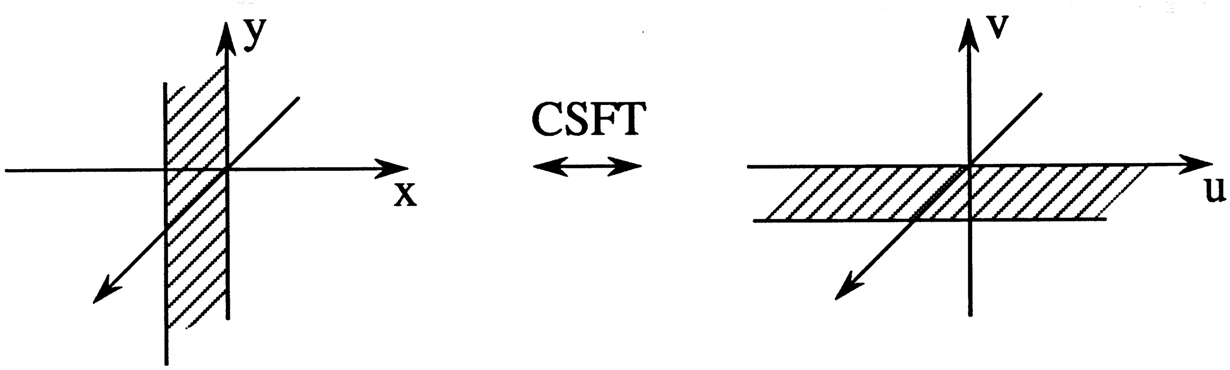
$\longleftrightarrow$



$$7.\ \ \mathrm{rect}(x) = \mathrm{rect}(x) \cdot 1 \quad \overset{\text{CSFT}}{\leftrightarrow} \quad \mathrm{sinc}(u)\ \delta(v)$$



$$8.\ \ \delta(x) = \delta(x) \cdot 1 \quad \overset{\text{CSFT}}{\leftrightarrow} \quad 1 \cdot \delta(v) = \delta(v)$$



Polar Coordinate CSFT

Polar coordinate transformation

spatial domain

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\tilde{f}(r,\theta) = f(x,y)$$

frequency domain

$$u = \rho\cos\phi$$
$$v = \rho\sin\phi$$
$$\tilde{F}(\rho,\phi) = F(u,v)$$

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\; e^{-j2\pi(ux+vy)}\; dxdy$$

$$ux + vy = \rho r \cos(\phi - \theta) \quad dxdy = rdrd\theta$$

Forward transform

$$\tilde{F}(\rho,\phi) = \int_{0}^{2\pi} \int_{0}^{\infty} \tilde{f}(r,\theta)\; e^{-j2\pi\rho r\cos(\phi - \theta)} rdrd\theta$$

Inverse transform

$$\tilde{f}(r,\theta) = \int_{0}^{2\pi} \int_{0}^{\infty} \tilde{F}(\rho,\phi)\; e^{j2\pi\rho r\cos(\phi-\theta)} \rho d\rho d\phi$$

Transform Relations

8. rotation

$$\tilde{f}(r,\ \theta + \theta_0) \overset{\text{CSFT}}{\longleftrightarrow} \tilde{F}(\rho,\ \phi + \theta_0)$$

9. circular symmetry

$$\tilde{f}(r,\theta) = \tilde{f}_0(r) \Leftrightarrow \tilde{F}(\rho,\phi) = \tilde{F}_0(\rho)$$

Fourier-Bessel (Zero Order Hankel) Transform Pair

Forward transform

Assume $\tilde{f}(r,\theta) = \tilde{f}_0(r)$

$$\tilde{F}(\rho,\phi) = \int_0^{\infty} \tilde{f}_0(r) \int_0^{2\pi} e^{-j2\pi\rho r\cos(\phi - \theta)}\, d\theta\, r dr$$

$$= 2\pi \int_0^{\infty} \tilde{f}_0(r)\, J_0(2\pi\rho r) r dr$$

(Bessel function of 1st kind, order 0)

$$= \tilde{F}_0(\rho)$$

Inverse transform

$$\tilde{f}_0(r) = 2\pi \int_0^{\infty} \tilde{F}_0(\rho)\, J_0(2\pi\rho r)\rho d\rho$$