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HW#8

1. Find expressions for the N point DFT's of the following signals.

a. $x(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n=1,\ldots,N-1 \end{cases}$

By definition,

$$X_N(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}}$$

$$X_N(K) = x(0)e^{-j\frac{2\pi K(0)}{N}} + x(1)e^{-j\frac{2\pi K(1)}{N}} + \ldots + x(N-1)e^{-j\frac{2\pi K(N-1)}{N}}$$

$$X_N(K) = 1e^0 + 0 + \ldots + 0$$

$$\boxed{X_N(K) = 1}$$

b. $x(n) = (-1)^n$, $n = 0, \ldots, N-1$, (Assume N is even)

$$X_N(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}}$$

$$= \sum_{n=0}^{N-1} (-1)^n e^{-j\frac{2\pi K n}{N}} = \sum_{n=0}^{N-1} \left(-e^{-j\frac{2\pi K}{N}}\right)^n$$

$$= \frac{1 - \left(-e^{-j\frac{2\pi K}{N}}\right)^N}{1 - \left(-e^{-j\frac{2\pi K}{N}}\right)} = \frac{1 - (-1)^N \left(e^{-j\frac{2\pi K}{N}}\right)^N}{1 + e^{-j\frac{2\pi K}{N}}}$$

$$= \frac{1 - e^{-j2\pi K}}{1 + e^{-j\frac{2\pi K}{N}}} = \begin{cases} \lim_{K \to \frac{N}{2}} \dfrac{1 - e^{-j2\pi K}}{1 + e^{-j\frac{2\pi K}{N}}} & K = \frac{N}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{X_N(K) = \begin{cases} N & K = \frac{N}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } K \in \{0, \ldots, N-1\}}$$

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1\. c. $x(n) = e^{j2\pi n K_0/N}$, $n = 0, \dots, N-1$ where $K_0 \in \{0, 1, \dots, N-1\}$

$$X_N(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nK}{N}}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi n K_0}{N}} e^{-j\frac{2\pi nK}{N}}$$

$$= \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi}{N}(K-K_0)}\right)^n$$

$$= \frac{1 - \left(e^{-j\frac{2\pi(K-K_0)}{N}}\right)^N}{1 - e^{-j\frac{2\pi(K-K_0)}{N}}} = \frac{1 - e^{-j2\pi(K-K_0)}}{1 - e^{-j\frac{2\pi(K-K_0)}{N}}}$$

$$X_N(K) = \begin{cases} \lim_{K \to K_0} \dfrac{1 - e^{-j2\pi(K-K_0)}}{1 - e^{-j\frac{2\pi(K-K_0)}{N}}} & \text{when } K = K_0 \\ 0 & \text{otherwise} \end{cases}$$

Using L'Hospital's Theorem

$$\lim_{K \to K_0} \frac{1 - e^{-j2\pi(K-K_0)}}{1 - e^{-j\frac{2\pi(K-K_0)}{N}}} = \lim_{K \to K_0} \frac{j2\pi e^{-j2\pi(K-K_0)}}{j\frac{2\pi}{N} e^{-j\frac{2\pi(K-K_0)}{N}}} = \frac{j2\pi}{\left(j\frac{2\pi}{N}\right)} = N$$

$$\boxed{X_N(K) = \begin{cases} N & \text{when } K = K_0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } K \in \{0, \dots, N-1\}}$$

d. $x(n) = \cos(2\pi K_0 n/N)$, $n = 0, \dots, N-1$, $K_0 \in \{0, 1, \dots, N-1\}$

$$X_N(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi Kn}{N}} = \sum_{n=0}^{N-1} \cos(2\pi K_0 n/N) e^{-j\frac{2\pi Kn}{N}}$$

$$X_N(K) = \sum_{n=0}^{N-1} \left(\frac{1}{2}\right)\left(e^{j\frac{2\pi K_0 n}{N}} + e^{-j\frac{2\pi K_0 n}{N}}\right) e^{-j\frac{2\pi Kn}{N}}$$

because $\cos x = \frac{1}{2}(e^{x} + e^{-x})$

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1.d. Futhermore, $e^{-j2\pi K_o n/N} = (1)^n e^{-j2\pi K_o n/N}$

$$= e^{j2\pi n} e^{-j2\pi K_o n/N}$$

$$= \left(e^{j\frac{2\pi N n}{N}}\right)\left(e^{-j\frac{2\pi K_o n}{N}}\right)$$

$$= e^{j\frac{2\pi(N-K_o)n}{N}} \quad \text{and } (N-K_o)\in\{0,1,\ldots,N-1\}$$

$$X_N(K) = \sum_{n=0}^{N-1}\left(\frac{1}{2}\right)\left(e^{j\frac{2\pi K_o n}{N}} + e^{-j\frac{2\pi K_o n}{N}}\right)e^{-j\frac{2\pi K n}{N}}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} e^{j\frac{2\pi K_o n}{N}} e^{-j\frac{2\pi K n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} e^{j\frac{2\pi(N-K_o)n}{N}} e^{-j\frac{2\pi K n}{N}}$$

$$\boxed{X_N(K) = \begin{cases} \frac{1}{2} & \text{for } K = K_o \text{ or } N-K_o, \text{ and } K\in\{0,\ldots,N-1\} \\ 0 & \text{otherwise} \end{cases}}$$

2. a) Show that $y(n)$ is periodic with period N

$$y(n) = \sum_{k=0}^{N-1} e^{j2\pi kn/N}$$

$$y(n) = \frac{1 - e^{j2\pi n}}{1 - e^{j\frac{2\pi n}{N}}}$$

$$y(n) = \begin{cases} N & \text{whenever } n = mN \text{ and } m \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

$$y(n) = N \sum_{m=-\infty}^{\infty} \delta(n - mN)$$

$$y(n+N) = N \sum_{m=-\infty}^{\infty} \delta(n+N - mN) = N \sum_{m=-\infty}^{\infty} \delta(n - (m-1)N)$$

Let $l = m-1$, then

$$y(n+N) = N \sum_{l=-\infty}^{\infty} \delta(n - lN) = y(n)$$

Thus, $y(n)$ is periodic with period N.

b) From part (a) $y(n) = N \sum_{m=-\infty}^{\infty} \delta(n - mN)$

If $n = 0$, then $y(0) = N \sum_{m=-\infty}^{\infty} \delta(-mN)$

and $m = 0$ is the only value for m which results in a nonzero value for $\delta(-mN)$.

$\Rightarrow y(0) = N\delta(0 - (0)N) = N\delta(0) = N$

i.e. $y(0) = N$.

c) From part (a), $y(n) = N \sum_{m=-\infty}^{\infty} \delta(n - mN)$

$\delta(n - mN) = 1$ whenever $n = mN$ or $m = \frac{n}{N}$ is an integer

if $n \in \{1, \ldots, N-1\}$, then there is no $m \in \mathbb{N}$ so that $n - mN = 0 \Rightarrow \delta(n - mN) = 0 \;\forall m \in \mathbb{N}$. Thus,

$y(n) = 0 \quad \forall n \in \{1, \ldots, N-1\}$

2. d) $DFT^{-1}\{DFT\{x(n)\}\} = DFT^{-1}\{\sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi Km}{N}}\}$

$$= \frac{1}{N}\sum_{K=0}^{N-1}\left(\sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi Km}{N}}\right) e^{j\frac{2\pi Kn}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m) \sum_{K=0}^{N-1} e^{-j\frac{2\pi(m-n)K}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left(N\sum_{d=-\infty}^{\infty} \delta((m-n)-dN)\right)$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} N x(m)\,\delta(m-n) \quad ;$$

since the sum over m goes from 0 to N-1

$$\Rightarrow DFT^{-1}\{DFT\{x(n)\}\} = \frac{1}{N}(N)\sum_{m=0}^{N-1} x(m)\,\delta(m-n)$$

$$= \sum_{m=0}^{N-1} x(m)\,\delta(m-n)$$

$$= x(n) \quad \text{for} \quad n \in \{0, \ldots, N-1\}$$

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3. a) $w_L(n) = \begin{cases} 1 & \text{for } 0 \le n < L \\ 0 & \text{otherwise} \end{cases}$ for $L \le N$

b) $x(n) = e^{j\omega_0 n} w_L(n)$ for $L \le N$ and $0 < \omega_0 < \pi$

a) $$W_N(K) = \sum_{n=0}^{N-1} w_L(n) e^{-j\frac{2\pi K n}{N}}$$

$$= \frac{1 - e^{-j2\pi K \frac{L}{N}}}{1 - e^{-j\frac{2\pi K}{N}}}$$

$$= \frac{e^{-j\frac{\pi K L}{N}} \left(e^{j\frac{\pi K L}{N}} - e^{-j\frac{\pi K L}{N}}\right)}{e^{-j\frac{\pi K}{N}} \left(e^{j\frac{\pi K}{N}} - e^{-j\frac{\pi K}{N}}\right)}$$

$$\boxed{W_N(K) = e^{-j\frac{\pi K(L-1)}{N}} \frac{\sin\left(\frac{\pi K L}{N}\right)}{\sin\left(\frac{\pi K}{N}\right)}}$$

b) $$X_N(K) = \sum_{n=0}^{N-1} e^{j\omega_0 n} w_L(n) e^{-j\frac{2\pi K n}{N}}$$

$$= \sum_{n=0}^{N-1} w_L(n) e^{-j\left(\frac{2\pi}{N}\left(K - \frac{N\omega_0}{2\pi}\right)\right)n}$$

$$= W_N\left(K - \frac{N\omega_0}{2\pi}\right)$$

$$\boxed{X_N(K) = e^{-j\frac{\pi\left(K - \frac{N\omega_0}{2\pi}\right)}{N}} \frac{\sin\left(\frac{\pi L}{N}\left(K - \frac{N\omega_0}{2\pi}\right)\right)}{\sin\left(\frac{\pi}{N}\left(K - \frac{N\omega_0}{2\pi}\right)\right)}}$$