ECE 438 Fall 2007

HW #7

1.

Def: A system is BIBO stable if $\forall$ bounded $x(n)$, (i.e. $\exists$ some $M_x \in \mathbb{R}$ such that $|x(n)| \leq M_x < \infty \ \forall n$), the output is also bounded.

Def. A sequence $h(K)$ is absolutely summable if the summation of the absolute value of the terms of the sequence converges,

$$\sum_{K=-\infty}^{\infty} |h(K)| < \infty.$$

Prove: The LTI system is BIBO stable if and only if $h(n)$ is absolutely summable.

Let P be the case that the system is BIBO.
Let Q be the case where $h(n)$ is absolutely summable.

To show that P is true if and only if Q is true, we must show ① P implies Q and ② Q implies P

① P implies Q
To show $P \Rightarrow Q$, we prove $\bar{Q} \Rightarrow \bar{P}$
So we assume that $h(n)$ is not absolutely summable and show that the system is not BIBO

ECE 438

Fall 2007

HW#7

1. (1) $P \Rightarrow Q$ or $\bar{Q} \Rightarrow \bar{P}$

Assume that $h(n)$ is not absolutely summable. Let $x(n) \doteq \text{sign}(h(-n))$. Then, $x(n)$ is a bounded signal. Furthermore, since the system is LTI

$$y(n) = x(n) * h(n) \quad \text{and}$$

$$y(n) = \sum_{K=-\infty}^{\infty} x(n-K)\, h(K)$$

by the definition for convolution in the discrete time domain. Thus,

$$y(n) = \sum_{K=-\infty}^{\infty} \text{sign}(h(-(n-K)))\, h(K)$$

$$y(n) = \sum_{K=-\infty}^{\infty} \text{sign}(h(K-n))\, h(K)$$

by definition of $x(n)$. And at $n=0$,

$$y(0) = \sum_{K=-\infty}^{\infty} |h(K)|$$

But since $h(n)$ is not absolutely summable $y(0)$ is infinite. Thus, $y(n)$ is not bounded $\forall n \in \mathbb{N}$ and every bounded $x(n)$. Thus, by definition $h(n)$ not absolutely summable implies the system is not BIBO stable. And the system being BIBO stable implies that the impulse response is absolutely summable.

But we are not done yet, we must also show that Q implies P.

HW#7

1.② $Q \Rightarrow P$

To show Q implies P, we assume that h(n) is absolutely summable and show that the system is BIBO. Let x(n) be a bounded input. Since the system is LTI,

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\,h(k).$$

$$\Rightarrow |y(n)| = \left|\sum_{k=-\infty}^{\infty} x(n-k)\,h(k)\right|$$

$$\Rightarrow |y(n)| \le \sum_{k=-\infty}^{\infty} |x(n-k)h(k)|$$

$$\Rightarrow |y(n)| \le \sum_{k=-\infty}^{\infty} |x(n-k)|\,|h(k)|$$

Because x(n) is bounded, $\exists$ M_x such that

$$|x(n)| \le M_x < \infty \quad \forall\, n \in \mathbb{N}$$

Therefore, $|y(n)| \le \sum_{k=-\infty}^{\infty} |x(n-k)|\,|h(k)| \le \sum_{k=-\infty}^{\infty} M_x\,|h(k)|$.

And since h(n) is absolutely summable, $\exists$ a M such that

$$\sum_{k=-\infty}^{\infty} |h(k)| \le M < \infty .$$

Thus, $\sum_{k=-\infty}^{\infty} M_x\,|h(k)| = M_x \sum_{k=-\infty}^{\infty} |h(k)| \le M_x M < \infty$, and

$$|y(n)| \le \sum_{k=-\infty}^{\infty} M_x\,|h(k)| \le M_x M < \infty .$$

Therefore, if h(n) is absolutely summable,

HW#7

1(a) then the system is BIBO.

And since $P \Rightarrow Q$ and $Q \Rightarrow P$, P is true if and only if Q is true. That is, the LTI system is BIBO if and only if $h(n)$, the impulse response, is absolutely summable.

HW#7

2. Let the impulse response $h(n)$ have a Z-transform $H(z)$ with a region of convergence Ω. Prove that $h(n)$ is absolutely summable if and only if $1 \in \Omega$

Let P be the case that $h(n)$ is absolutely summable and let Q be the case that $1 \in \Omega$.

To show that P is true if and only if Q is true. We must show 1 P implies Q and 2 Q implies P.

1 P implies Q

Assume $h(n)$ is absolutely summable and show that $1 \in \Omega$.

Since $h(n)$ is absolutely summable, $\exists$ a scalar M such that

$$\sum_{n=-\infty}^{\infty} |h(n)| \leq M < \infty$$

By definition of the z-transform, the sum, $\sum_{n=-\infty}^{\infty} h(n) z^{-n}$, converges and is equal to $H(z)$ $\forall z \in \Omega$.

Thus,

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right|$$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |h(n)| \, |z^{-n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |h(n)| \, |z|^{-n}$$

ECE 438 Fall 2007

HW #7

2. 1 If $z=1$, then

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|\, 1^{-n}$$

$$\Rightarrow \quad |H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| \leq M < \infty$$

Thus, by definition of the z-transform and conditions for its convergence, $1 \in \Omega$. That is, $P \Rightarrow Q$

2 Q implies P
Assume that $1 \in \Omega$ and show that $h(n)$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h(n) z^{-n}| < \infty \qquad \text{for } z \in \Omega.$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h(n)(1^{-n})| < \infty \qquad \text{since } 1 \in \Omega$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$n=-\infty$

Thus, if $1 \in \Omega$, the impulse response is absolutely summable. i.e. $Q \Rightarrow P$

Since we have shown $P \Rightarrow Q$ and $Q \Rightarrow P$, P is true if and only if Q is true.

ECE 438 Fall 2007

HW#7

3. $H(z) = \mathcal{Z}\{h(n)\}$, $H(z)$ is a rational function and the Z-transform has ROC Ω

i) If $\{z : |z| > a\} \subset \Omega \quad \forall a \in (0, \infty)$, then

$$\mathbb{C} - \{0\} \subset \Omega \qquad \text{i.e: } z=0 \text{ is a pole}$$

$\Rightarrow$ $\boxed{h(n) \text{ is right-sided}}$

ii) If $\{z : |z| < a\} \subset \Omega \quad \forall a \in (0, \infty)$, then

$$\{z : |z| < \infty\} \subset \Omega \qquad \text{i.e. } z = \infty \text{ is a pole}$$

$\Rightarrow$ $\boxed{h(n) \text{ is left-sided!}}$

iii) If $0 \in \Omega \Rightarrow h(n)$ is anticausal
iv) If $\infty \in \Omega \Rightarrow h(n)$ is causal

Note: If the system is causal, it is also right-sided. And if the system is anticausal, then it is also left-sided.

ECE 438 Fall 2007

HW #7

4. $$H(z) = \frac{z^2 - z}{(z^2 - 1/4)(z - 1/2)}$$

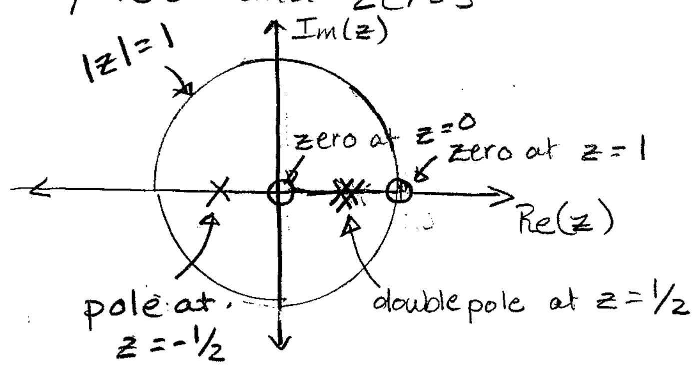
a) Find the poles and zeros for $H(z)$.

single zeros at $z = 1$ and $z = 0$ from $(z^2 - z) = 0$

double poles at $z = 1/2$ and a single pole at $z = -1/2$ from solving $(z^2 - 1/4)(z - 1/2) = 0$.

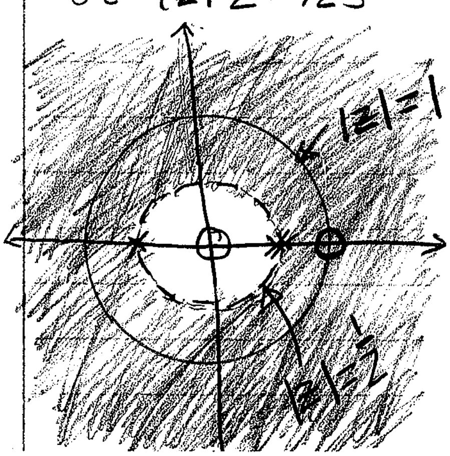
b) Sketch the poles and zeros and the possible ROC's.

poles and zeros

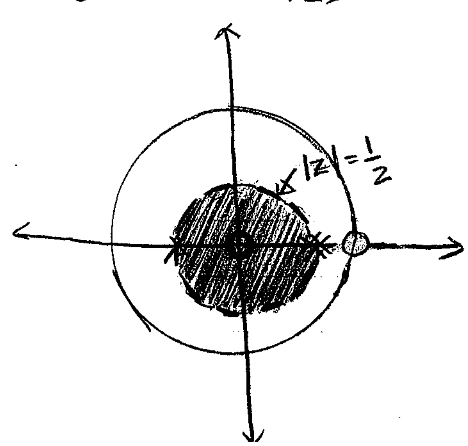


Possible ROCs shown in shaded region

Option 1:
$\Omega = \{z \mid z > 1/2\}$



Option 2: $|z| < 1/2$
$\Omega = \{z \mid z < 1/2\}$



ECE 438

Fall 2007

HW#7

4. c) ROC : $|z| > \frac{1}{2}$, the $h(n)$ is causal and $h(n)$ is right-sided.

ROC: $|z| < \frac{1}{2}$, the system is anticausal and $h(n)$ is left-sided

d) ROC : $|z| > \frac{1}{2}$. The system is stable since ROC contains $|z| = 1$

ROC: $|z| < \frac{1}{2}$. The system is unstable because the ROC does not contain $|z| = 1$

ECE 438 Fall 2007

HW#7

4. e) $$H(z) = \frac{z^2 - z}{(z^2 - 1/4)(z - 1/2)}$$

$$= \frac{z^{-3}(z^2 - z)}{(z^2 - 1/4)(z - 1/2)}$$

$$= \frac{z^{-1} - z^{-2}}{(1 - (1/4)z^{-2})(1 - (1/2)z^{-1})}$$

$$= \frac{z^{-1} - z^{-2}}{(1 + (1/2)z^{-1})(1 - (1/2)z^{-1})(1 - (1/2)z^{-1})}$$

$$= \frac{A}{1 + (1/2)z^{-1}} + \frac{B}{1 - (1/2)z^{-1}} + \frac{C}{(1 - (1/2)z^{-1})^2}$$

To find C

$z = 1/2$ $\Rightarrow$ $$\frac{(1/2)^{-1} - (1/2)^{-2}}{1 + 1/2 (1/2)^{-1}} = C$$

$$\frac{-2}{2} = C \Rightarrow C = -1$$

To find A

$z = -1/2$ $\Rightarrow$ $$\frac{(-1/2)^{-1} - (-1/2)^{-2}}{(1 - (1/2)(-1/2)^{-1})^2} = A$$

$$\frac{-2 - 4}{(1 - (-1))^2} = A \Rightarrow A = -\frac{3}{2}$$

To find B

$z = 1$

$$0 = A(1 - (1/2)(1)^{-1})^2 + B(1 - (1/2)(1)^{-1})(1 + (1/2)(1)^{-1}) + C(1 + (1/2)(1)^{-1})$$

$$0 = -\frac{3}{2}(1/2)^2 + B(1/2)(3/2) + -1(3/2)$$

$$\frac{3}{8} + \frac{12}{8} = \frac{6B}{8}$$

$$\Rightarrow B = \frac{5}{2}$$

4. e) Check: $\frac{-3}{2}(1-(1/2)z^{-1})^2 + \frac{5}{2}(1-(1/2)z^{-1})(1+(1/2)z^{-1}) - 1(1+(1/2)z^{-1})$

$$= \frac{-3}{2}(1 - z^{-1} + (1/4)z^{-2}) + \frac{5}{2}(1-(1/4)z^{-2}) - 1(1+1/2\, z^{-1})$$

$$= \frac{-3}{2} + \frac{3}{2}z^{-1} + \frac{-3}{8}z^{-2} + \frac{5}{2} - \frac{5}{8}z^{-2} - 1 - \frac{1}{2}z^{-1}$$

$$= \left(\frac{-3}{2} + \frac{5}{2} - 1\right) + \left(\frac{3}{2} - \frac{1}{2}\right)z^{-1} + \left(\frac{-3}{8} - \frac{5}{8}\right)z^{-2}$$

$$= z^{-1} - z^{-2} \quad \checkmark$$

$$H(z) = \frac{\frac{-3}{2}}{1+(1/2)z^{-1}} + \frac{\frac{5}{2}}{1-(1/2)z^{-1}} + \frac{-1}{(1-(1/2)z^{-1})^2}$$

$$= \frac{-\frac{3}{2}}{1+(1/2)z^{-1}} + \frac{\frac{5}{2}}{1-(1/2)z^{-1}} - 2z\left(\frac{(1/2)z^{-1}}{(1-(1/2)z^{-1})^2}\right)$$

ROC: $\Omega_1 = \{z \in \mathbb{C} \mid |z| > \frac{1}{2}\}$

$$h(n) = \frac{-3}{2}\left(\frac{-1}{2}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{2}\right)^n u(n) - 2(n+1)\left(\frac{1}{2}\right)^{n+1} u(n+1)$$

$$\boxed{h(n) = \frac{-3}{2}\left(\frac{-1}{2}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{2}\right)^n u(n) - (n+1)\left(\frac{1}{2}\right)^n u(n+1)}$$

ROC: $\Omega = \{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}$

$$h(n) = \frac{3}{2}\left(\frac{-1}{2}\right)^n u(-n-1) - \frac{5}{2}\left(\frac{1}{2}\right)^n u(-n-1) + 2(n+1)\left(\frac{1}{2}\right)^{n+1} u(-(n+1)-1)$$

$$\boxed{h(n) = \frac{3}{2}\left(\frac{-1}{2}\right)^n u(-n-1) - \frac{5}{2}\left(\frac{1}{2}\right)^n u(-n-1) + (n+1)\left(\frac{1}{2}\right)^n u(-n-2)}$$

ECE 438 Fall 2007

HW # 7

4. f)

$$H(z) = \frac{z^2 - z}{(z^2 - 1/4)(z - 1/2)}$$

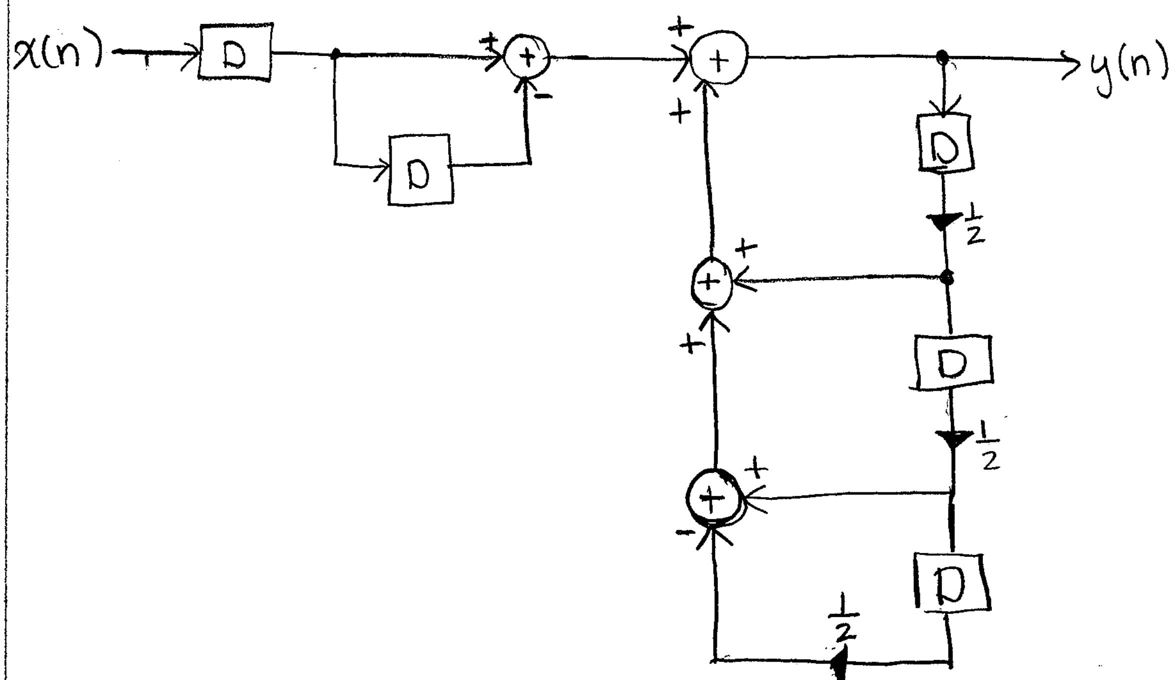
$$= \frac{z^{-3}\,(z^2 - z)}{z^{-3}(z^2 - 1/4)(z - 1/2)}$$

$$= \frac{z^{-1} - z^{-2}}{z^{-3}(z^3 - (1/2)z^2 - (1/4)z + 1/8)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - z^{-2}}{1 - (1/2)z^{-1} - (1/4)z^{-2} + (1/8)z^{-3}}$$

$$Y(z)(1 - (1/2)z^{-1} - (1/4)z^{-2} + (1/8)z^{-3}) = X(z)(z^{-1} - z^{-2})$$

$$\boxed{y(n) = 1/2\, y(n-1) + 1/4\, y(n-2) - 1/8\, y(n-3) + x(n-1) - x(n-2)}$$



12

ECE 438 Fall 2007

HW#7

5. $$y(n) = ay(n-1) + x(n) - x(n-1)$$

a) $$Y(z) = az^{-1}Y(z) + X(z) - z^{-1}X(z)$$

$$H(z) = \frac{1-z^{-1}}{1-az^{-1}} = \frac{z-1}{z-a}$$

$\boxed{\text{poles at } z=a \quad \text{and} \quad \text{zeros at } z=1}$

b) ROC $|z|>a$

$$H(z) = \frac{1}{1-az^{-1}} - \frac{z^{-1}}{1-az^{-1}}$$

$$\Rightarrow \boxed{h(n) = a^n u(n) - a^{(n-1)} u(n-1)}$$

$\boxed{\text{The system is stable for } |a|<1.}$

c) ROC $|z|<a$

$$H(z) = \frac{1}{1-az^{-1}} - \frac{z^{-1}}{1-az^{-1}}$$

$$\Rightarrow h(n) = -a^n u(-n-1) + a^{n-1} u(-(n-1)-1)$$

$$\boxed{h(n) = -a^n u(-n-1) + a^{n-1} u(n)}$$

$\boxed{\text{The system is stable if ROC contains } |z|=1 \Rightarrow |a|>1.}$

ECE 438 Fall 2007

HW #7

6. $$y(n) = x(n) - x(n-8) + y(n-1)$$

a) $$Y(z) = X(z) - z^{-8}X(z) + z^{-1}Y(z)$$

$$Y(z)(1 - z^{-1}) = (1 - z^{-8})X(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-8}}{1 - z^{-1}}}$$

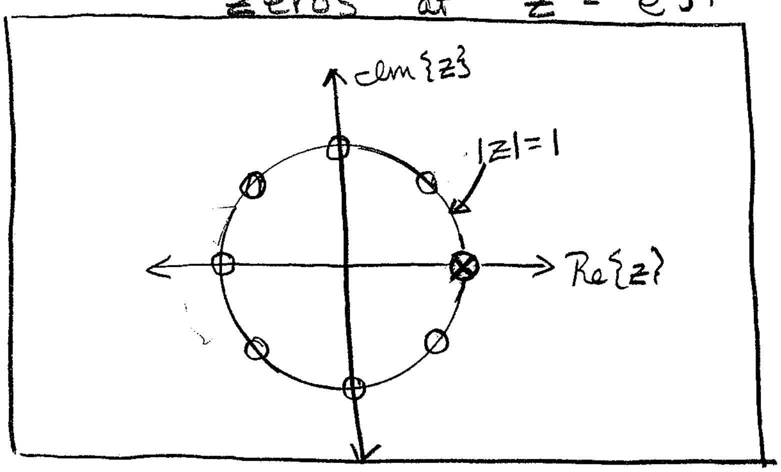
b) Pole when $1 - z^{-1} = 0 \Rightarrow z = 1$ is the pole

Zeros when $1 - z^{-8} = 0 \Rightarrow z^{-8} = 1$

$$z^{-8} = e^{j(0 + 2\pi K)}$$

$$z = e^{j(0+2\pi K)/(-8)}$$

zeros at $z = e^{j\frac{\pi}{4}K}$ $K \in \{0, 1, 2, 3, 4, 5, 6, 7\}$



c) Possible ROCs include $|z| > 1$ and $|z| < 1$

ROC: $|z| > 1 \Rightarrow \boxed{h(n) = u(n) - u(n-8)}$

ROC: $|z| < 1 \Rightarrow h(n) = -1^n u(-n-1) + 1^{(n-8)} u(-(n+8) - 1)$

$$\boxed{h(n) = -u(-n-1) + u(-n+7)}$$

ECE 438 Fall 2007

HW#7

6 d) The filter is FIR because $h(n)$ has finite length for both ROC's and the pole of the transfer function is canceled by the zero at 1.